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MA637 Numerical Analysis & Computing

Winter Semester 24/25

13.01.2025

Assigment 1

Assignment and programming exercises must be completed in groups of 4 members and submitted via Google Classroom by 20.01.2025 at 23:59. If the assignment is not prepared using LaTeX, a clear scanned copy of the handwritten work must be uploaded. Ensure that the names and enrollment numbers of all group members are clearly written on the submission. Late submissions will not be accepted.

Question 1: [**Vandermonde Matrix**]

Let $\mathbf{V}$ denote the Vandermonde matrix given by

$$\mathbf{V} = \begin{bmatrix} x_0^n & x_0^{n-1} & \dots & x_0 & 1 \\ x_1^n & x_1^{n-1} & \dots & x_1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_n^n & x_n^{n-1} & \dots & x_n & 1 \end{bmatrix}.$$

Show that

$$\det(\mathbf{V}) = \prod_{i=0}^{n-1} \prod_{j=i+1}^{n} (x_i - x_j).$$

Question 2: [**Lagrange Interpolation**]

Let $f(x) = e^x$, for $0 \le x \le 2$.

a) Approximate $f(0.25)$ using linear interpolation with $x_0 = 0$ and $x_1 = 0.5$.

b) Approximate $f(0.75)$ using linear interpolation with $x_0 = 0.5$ and $x_1 = 1$.

c) Approximate $f(0.25)$ and $f(0.75)$ by using the second interpolating polynomial with $x_0 = 0$, $x_1 = 1$ and $x_2 = 2$.

d) Which approximations are better and why?

Question 3: [**Bernstein Polynomials**]

The Bernstein polynomial of degree n for $f \in \mathcal{C}[0,1]$ is given by

$$B_n(x) = \sum_{k=0}^{n} {}^n\mathrm{C}_k f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}.$$

a) Find $B_3(x)$ for the function

i) $f(x) = x$

ii) $f(x) = 1$.

b) Show that for each $k \leq n$

$$^{n-1}\mathrm{C}_{k-1} = \left(\frac{k}{n}\right) {}^{n}\mathrm{C}_k.$$

c) Use **b)** and the fact, from **ii)** in **a)** that

$$1 = \sum_{k=0}^{n} {}^{n}\mathrm{C}_k x^k (1-x)^{n-k}, \quad \text{for each } n,$$

to show that for $f(x) = x^2$,

$$B_n(x) = \left(\frac{n-1}{n}\right) x^2 + \frac{1}{n} x.$$

Question 4: [Programming Exercise]

The goal of this exercise is to understand polynomial interpolation using the Vandermonde Interpolation algorithm and analyze its accuracy for different dataset sizes. Create a data set of points $\{(x_i, f_i)\}_{i=0}^{n}$ using the functions

$$f_1(x) = x^2, \quad f_2(x) = \sin(x), \quad \text{and} \quad f_3(x) = \frac{1}{1+2x^2},$$

in the interval $[0, 2]$.

a) Write a code for polynomial interpolation using the Vandermonde Interpolation algorithm.

b) Use $n = 5, 25, 50, 100$.

c) Check your solution by evaluating the interpolated polynomial at $x = \pi/2$. Compute the error ($\texttt{Error} = |\texttt{Interpolated Value} - \texttt{Exact Value}|$) and comment for what values of n in **b)** do you observe the least error. Justify your answer.

Hint: In this programming exercise, you would require the following programming concepts:

a) `for` loops.

b) `numpy` library:

- **i)** `linspace`: For creating equidistant points.
- **ii)** `size`: To get the number of elements in the numpy array.
- **iii)** `linalg.solve`: The solve the system of equation $\mathbf{Ax} = \mathbf{b}$.

c) Function declaration using `def`.