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MA637 Numerical Analysis & Computing

Winter Semester 24/25

14.02.2025

Assigment 5

Assignment and programming exercises must be completed in groups of 4 members and submitted via Google Classroom by 24.02.2025 at 23:59. If the assignment is not prepared using LaTeX, a clear scanned copy of the handwritten work must be uploaded. Ensure that the names and enrollment numbers of all group members are clearly written on the submission. Late submissions will not be accepted.

Question 1: [**Special Matrices**]

Let $\mathbf{A}$ be a 3×3 matrix given by

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & \alpha \end{bmatrix}.$$

Find all the values of α for which $\mathbf{A}$ is:

a) Singular.

b) Strictly Diagonally Dominant.

c) Symmetric

d) Positive Definite

Question 2: [**Crout Factorisation**]

Let $\mathbf{Ax} = \mathbf{b}$ be a system of equations such that $\mathbf{A}$ is a $n \times n$ matrix. In the lecture, we studied the Doolittle factorization which factors the matrix $\mathbf{A} = \mathbf{LU}$ where $\mathbf{L}$ is a unit lower triangular matrix and $\mathbf{U}$ is an upper triangular matrix. We can also factorise this matrix as $\mathbf{A} = \hat{\mathbf{L}}\hat{\mathbf{U}}$ where $\hat{\mathbf{L}}$ is a lower triangular matrix and $\hat{\mathbf{U}}$ is an unit upper triangular matrix, i.e.,

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} \ell_{11} & 0 & \dots & 0 \\ \ell_{21} & \ell_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \ell_{n1} & \ell_{n2} & \dots & \ell_{nn} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & \dots & u_{1n} \\ 0 & 1 & \dots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

Use Crout factorization and forward/backward substitution to solve the following system of equations.

$$\begin{array}{ccccccc} 2x_1 & + & x_2 & & & = & 3 \\ x_1 & + & 2x_2 & + & x_3 & = & -2 \\ & & 2x_2 & + & 3x_3 & = & 0 \end{array}$$

Question 3: [**Cholesky Decomposition**]

Given a symmetric positive definite matrix $\mathbf{A}$, its Cholesky decomposition is defined as $\mathbf{A} = \mathbf{L}\mathbf{L}^\top$, where $\mathbf{L}$ is a lower triangular matrix. In the lecture notes, Algorithm 10 presents the Cholesky decomposition process.

a) Extend Algorithm 10 to solve the system of linear equations $\mathbf{A}\mathbf{x} = \mathbf{b}$.

b) Demonstrate that the computational complexities of the forward and backward substitution steps are:

- $n^2 + n$ operations for multiplication and division.
- $n^2 - n$ operations for addition and subtraction.

Note: Follow the structure used in the lecture notes when presenting the algorithm. Specifically, use explicit `for` loops to show summations. You may use either the `algorithm` package or simple `enumerate` or `itemize` structures.

Question 4: [**Programming Exercise**]

The goal of this exercise is to understand $\mathrm{LDL}^\top$ and analyze its accuracy for different system of equations. Consider the system of equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ where

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 2 & 1 \\ 0 & 3 & -1 & 1 \\ 2 & -1 & 6 & 3 \\ 1 & 1 & 3 & 8 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} -8 \\ -20 \\ -2 \\ 4 \end{bmatrix}.$$

a) Write a code for $\mathrm{LDL}^\top$ decomposition to solve the above system of equation to get the numerical solution $\mathbf{x}_\mathrm{N}$.

b) Modify the given code such that it also checks if the given matrix $\mathbf{A}$ is symmetric or not. Check your code for the following matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 8 \\ 1 & 8 & 4 \end{bmatrix}.$$

c) The matrix given in **a)** is also a symmetric positive definite matrix. Now, given the $\mathrm{LDL}^\top$ decomposition, we can also compute the Cholesky decomposition ($\hat{\mathbf{L}}\hat{\mathbf{L}}^\top$), where $\hat{\mathbf{L}} = \mathbf{L}\mathbf{D}^{1/2}$. Using this information, compute the Cholesky Decomposition.

Hint: In this programming exercise, you would require the following programming concepts:

a) `for` loops.

b) `if-else` structure.

c) Function declaration using `def`.

d) `numpy` library:

  i) `array`: For creating arrays and matrices.

  ii) `size`: To get the number of elements in the numpy array.

  iii) `zeros`: To create an array with entries as zero.

  iv) `identity`: To create the identity matrix.