Dr. A. Jha
M.Sc. A. Khan

MA637 Numerical Analysis & Computing

Winter Semester 24/25

03.04.2025

Assigment 8

Assignment and programming exercises must be completed in groups of 4 members and submitted via Google Classroom by 13.04.2025 at 23:59. If the assignment is not prepared using LaTeX, a clear scanned copy of the handwritten work must be uploaded. Ensure that the names and enrollment numbers of all group members are clearly written on the submission. Late submissions will not be accepted.

Question 1: [**Multistep Method**]

Consider the differential equation

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha.$$

a) Show that

$$y_i' = \frac{-3y_i + 4y_{i+1} - y_{i+2}}{2h} + \mathcal{O}\left(h^2\right).$$

b) The previous part suggests that there exists a difference method of the form

$$y_{i+2}^h = 4y_{i+1}^h - 3y_i^h - 2hf(t_i, y_i^h), \quad \text{for } \ i = 0, 1, \ldots, n-2.$$

Use this method to compute y_2^h for the given initial value problem

$$y' = 1 - y, \quad 0 \le t \le 1, \quad y(0) = 0,$$

with $h = 0.25$. Use the starting values $y_0^h = 0$ and $y_1^h = 0.0952$.

c) Analyze the method's stability, and convergence.

Question 2: [**Finite Difference Method**]

Consider the differential equation

$$y''(x) + y'(x) = 1, \quad a \le x \le b,$$

with the boundary conditions $y(a) = \alpha$ and $y(b) = \beta$. Divide the domain $[a, b]$ into $(n+2)$ equidistant subintervals with step size $h = (b-a)/(n+1)$. Using the central difference for $y''(x)$ and the forward difference for $y'(x)$, write down the system of equation $\mathbf{Ay} = \mathbf{b}$ where $\mathbf{A}$ is a $n \times n$ matrix.

Question 3: [**Existence and Uniqueness of Solution**]

Consider the linear boundary value problem

$$y''(x) = p(x)y'(x) + q(x)y(x) + r(x), \quad \text{for } \ a \le x \le b,$$

with $y(a) = \alpha$ and $y(b) = \beta$. Using central difference scheme we get a system of equation of the form

$$\mathbf{Ay} = \mathbf{b}.$$

Suppose that $p(x), q(x)$ and $r(x)$ are continuous on $[a, b]$. If $q(x) \geq 0$ on $[a, b]$, then the system of equation $\mathbf{Ay} = \mathbf{b}$ has a unique solution provided that $h < 2/L$ where $L = \max_{a \leq x \leq b} |p(x)|$.

Hint: You can use the following theorem

Theorem: Suppose that $\mathbf{A} = \{a_{ij}\}$ is a tridiagonal matrix with $a_{i,i-1}a_{i,i+1} \neq 0$, for each $i = 2, 3, \ldots, n-1$. If $|a_{11}| > |a_{12}|, |a_{ii}| \geq |a_{i,i-1}| + |a_{i,i+1}|$, for each $i = 2, 3, \ldots, n-1$, and $|a_{nn}| > |a_{n,n-1}|$, then $\mathbf{A}$ is nonsingular.

Question 4: [**Programming Exercise**]

This exercise aims to understand the Finite Difference Method for Linear Problem and analyze its accuracy. Consider the boundary value problem

$$\begin{aligned} y''(x) &= y'(x) + 2y(x) + \cos(x), \quad 0 \leq x \leq \frac{\pi}{2}, \\ y(0) &= -0.3, \quad y\left(\frac{\pi}{2}\right) = -0.1. \end{aligned}$$

a) Write a code to approximate the numerical solution $\{y_i^h\}$ using central finite difference method.

b) The analytical solution to the above problem is given by

$$y(x) = -\frac{1}{10}\Big(\sin(x) + 3\cos(x)\Big).$$

Compute the absolute error at each x_i for $h = \pi/8, \pi/16$ and $\pi/32$.

c) What do you observe as h decreases? How much is the reduction?

Hint: In this programming exercise, you would require the following programming concepts:

a) `for` loops.

b) `if-else` structure.

c) Function declaration using `def`.

d) `int`: Converts a float to integer type.

e) `numpy` library:

- **i)** `linspace`: For creating arrays of equal spacing.
- **ii)** `size`: For number of elements in an array.
- **iii)** `abs`: To get the absolute value.
- **iv)** `sin`: To get the sin function.
- **v)** `cos`: To get the cos function.
- **vi)** `concatenate`: Add value in the beginning or end of an array
- **vii)** `linalg.solve`: Solve system of equation $\mathbf{Ay} = \mathbf{b}$