Lecture 10: Differential Equations

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Outline

Differential Equations

Ordinary Differential Equations

Navier-Stokes equation

Let $\underline{u}(\underline{x},t)$ be the velocity field and $p(\underline{x},t)$ be the pressure. Then

$$\frac{\partial \underline{u}}{\partial t} - \nu \Delta \underline{u} \overset{\text{Non-linear}}{\boxed{+\, \underline{u}\cdot\nabla\underline{u}}} = -\nabla p + \underbrace{f(t,\underline{x})}_{\hookrightarrow\ \text{Sources.}}$$

$$\nabla\cdot\underline{u} = 0 \longrightarrow \text{Divergence free.}$$

Incompressible fluid.

constant density and volume under pressure.

$\mathbb{R}^3 \to \quad \underline{x} = (x,y,z) = (x_1,x_2,x_3) \;;\quad \underline{u} = (u_1,u_2,u_3)$

In $\mathbb{R}^3$; there is no existence and uniqueness of the solution.

Ordinary Differential Equations

$$\frac{dy}{dx} = y - x^2; \qquad y(0) = 2.$$

$$\frac{dy}{dx} = x - y^2, \qquad y(0) = 2$$

Differential Equations

Ordinary Differential Equations

$$y(x) = -\frac{ix^{3/2}\left\{-c_1 J_{-4/3}\left(\frac{2}{3}ix^{3/2}\right) + c_1 J_{2/3}\left(\frac{2}{3}ix^{3/2}\right) - 2J_{-2/3}\left(\frac{2}{3}ix^{3/2}\right)\right\} - c_1 J_{-1/3}\left(\frac{2}{3}ix^{3/2}\right)}{2x\left(J_{-1/3}\left(\frac{2}{3}ix^{3/2}\right) + J_{1/3}\left(\frac{2}{3}ix^{3/2}\right)\right)}$$

$$\underbrace{\frac{dy}{dt} = f(t,y).}_{\text{Initial value problem (IVP)}} \qquad a \le t \le b.$$

, Boundary value prob (BVP) (Later).

Theorem

Suppose

$$D = \{(t, y) : a \le t \le b, \ -\infty < y < \infty\},$$

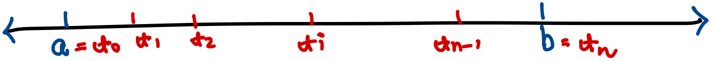
and let $f(t, y)$ be continuous on D. If f satisfies a Lipschitz condition in y on D, then the initial value problem has a unique solution $y(t)$ for $a \le t \le b$.

Lipschitz : There exists a $L>0$ st

$$|f(t, y_1) - f(t, y_2)| \le L|y_1 - y_2| , \ \forall (t, y_1), (t, y_2) \in D$$

One Step Method

$\frac{dy}{dt} = f(t,y)$; $a \le t \le b$; $y(a) = \alpha$



$y(a) = \alpha$

$[a,b] \to \{t_i\}_{i=0}^{n}$ s.t. $t_0 = a$; $t_n = b$; $\{t_i\} \to$ mesh/grid points

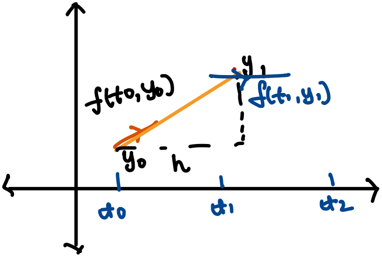
We assume that the mesh points are divided uniformly or equally;

$t_i = a + ih$; $i = 0, \dots, n$

$$\boxed{h} = \frac{b-a}{n} = t_{i+1} - t_i$$

Step size.

Euler Method



$y_0 = \alpha$

$y_1 = ?$

$y_2 = ?$

$\vdots$

$y_n = ?$

$$\frac{dy}{dt} = f(t,y)$$

Euler method

$$y_1 - y_0 = f(t_0,y_0)(t_1 - t_0) \Rightarrow \boxed{y_1 = y_0 + h f(t_0,y_0).}$$

Mathematically

$$y_1 = y(t_0+h) = y(t_0) + h \left.\frac{dy}{dt}\right|_{(t_0,y_0)} + O(h^2)$$

Same

$$\boxed{y_1 = y_0 + h f(t_0,y_0) + O(h^2)}$$

Euler Method

$$y^h_{i+1} = y^h_i + h f(t_i, y^h_i) \quad \text{for } i = 0, \cdots, n-1$$

$$y^h_0 = \alpha$$

Example : $y' = te^{3t} - 2y$, $0 \le t \le 1$, $y(0) = 0$

$h = 0.5$; y_0, y_1, y_2 ; $t_1 = 0.5$, $t_2 = 1$

$$y_0 = 0$$

$$y_1 = y_0 + hf(t_0, y_0) = 0 + 0.5 f(0, 0)$$

$$= 0.5\,[0e^0 - 2 \times 0] = 0$$

$$y_2 = y_1 + hf(t_1, y_1) = 0 + 0.5\,[f(0.5, 0)]$$

$$= 0.5\,[0.5 e^{1.5} - 0] = 1.12042.$$

Euler Method

Theorem

Suppose that f is continuous and satisfies a Lipschitz condition in y with Lipschitz constant L on the set

$$D = \{(t, y) : a \le t \le b,\ -\infty < y < \infty\}.$$

Assume further that the exact solution $y(t)$ of the initial value problem

$$y'(t) = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

is twice continuously differentiable on $[a, b]$, and that there exists a constant $M > 0$ such that

$$|y''(t)| \le M \quad \text{for all } t \in [a, b].$$

Let $\{y_i^h\}_{i=0}^n$ be the numerical approximations generated by Euler's method with step size $h = \frac{b-a}{n}$ and mesh points $t_i = a + ih$. Then, for each $i = 0, 1, 2, \ldots, n$, the global error satisfies

$$|y(t_i) - y_i^h| \le \boxed{\frac{hM}{2L}}\left(e^{L(t_i-a)} - 1\right). \longrightarrow \mathcal{O}(h).$$

h	$h^{d=2}$
0.1	0.01
0.01	1e-4
⋮	
1e-6	1e-12

d=2 Quadratically
d=1 Linear

$|y(t_i) - y_i| \sim \mathcal{O}(h)$;

Higher Order Taylor Method

$$y_1 = y(t_0+h) = y(t_0) + h\left.\frac{dy}{dt}\right|_{(t_0,y_0)} + O(h^2)$$

Defn: Local truncation error (LTE)

$$y_0 = \alpha$$

$$y_{i+1} = y_i + h\,\Phi(t_i, y_i, h) \quad \text{for } i = 0, 1, \dots, n-1$$

Euler $= f(t_i, y_i)$

Then LTE is

$$\tau_{i+1}(h) = \frac{y_{i+1} - (y_i + h\,\Phi(t_i, y_i, h))}{h} = \boxed{\frac{y_{i+1} - y_i}{h}} - \overbrace{\underbrace{\boxed{\Phi(t_i, y_i, h)}}_{\text{control}}}$$

for $i = 0, \dots, n-1$.

Remains same for high Order Taylor

Euler: $\tau_{i+1}(h) = \frac{h}{2} y''(\xi)$, $\xi \in (t_i, t_{i+1})$

Higher Order Taylor Method

$$y_{i+1} = y(t_i + h)$$

$$= y(t_i) + h \underbrace{\boxed{\left.\frac{dy}{dt}\right|_{t_i}}}_{f(t_i,\, y_i)} + \frac{h^2}{2} \underset{?}{\boxed{\left.\frac{d^2y}{dt^2}\right|_{t_i}}} + \dots + \frac{h^n}{n!} y_i^{(n)} + \frac{h^{n+1}}{(n+1)!} y^{(n+1)}(\xi_i)$$

for $\xi_i \in (t_i, t_{i+1})$

$$\frac{dy}{dt} = f(t,y); \quad \frac{d^2y}{dt^2} = \frac{d}{dt}(f(t,y)) \;; \quad \frac{d^n y}{dt^n} = \frac{d^{n-1}}{dt^{n-1}}(f(t,y))$$

<u>Higher-order</u>

$$y_0 = \alpha$$

$$y_{i+1} = y_i + h\,T^{(n)}(t_i, y_i) \qquad \text{for } i = 0, \dots, n-1$$

$$T^{(n)}(t_i, y_i) = f(t_i, y_i) + \frac{h}{2} \left.\frac{d}{dt}(f(t,y))\right|_i + \dots + \frac{h^{n-1}}{n!} \left. f^{(n-1)}(t, y(t)) \right|_i$$

Higher Order Taylor Method

Example: $y' = y - t^2 + 1$; $0 \le t \le 2$; $y(0) = 0.5$

$$f(t,y) = y - t^2 + 1$$

$T^{(2)}, T^{(3)}$:

$$\underbrace{\frac{d}{dt} f(t,y)}_{T^{(2)}} = \frac{d}{dt}(y - t^2 + 1) = \frac{dy}{dt} - 2t = f(t,y) - 2t$$

$$= y - t^2 + 1 - 2t$$

$$T^{(3)} \Rightarrow \frac{d}{dt}\left(\frac{d}{dt} f(t,y)\right) = \frac{d}{dt}\left(y - t^2 + 1 - 2t\right) = \frac{dy}{dt} - 2t - 2$$

$$= f(t,y) - 2t - 2 = y - t^2 + 1 - 2t - 2$$

$$= y - 2t - t^2 - 1.$$

$$T^{(2)}(t_i, y_i^h) = f(t_i, y_i^h) + \frac{h}{2} f'(t_i, y_i^h)$$

$$= y_i - t_i^2 + 1 + \frac{h}{2}\left(y_i - t_i^2 + 1 - 2t_i\right) =$$

Higher Order Taylor Method

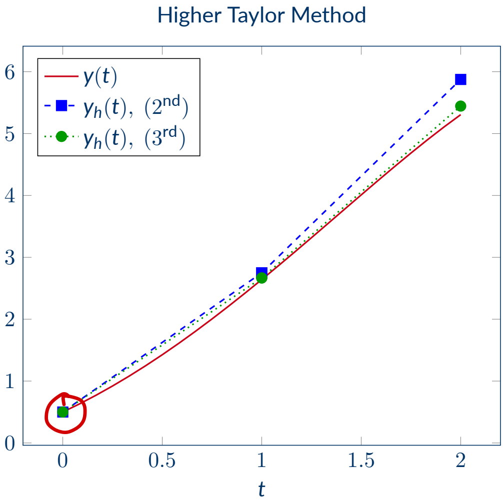


Figure 1: Taylor methods of order 2 and 3.

Higher Order Taylor Method

Theorem

If the Taylor method of order n is used to approximate the solution of

$$y'(t) = f(t, y(t)), \quad a \le t \le b, \quad y(a) = \alpha,$$

with step size h, and if $y \in \mathcal{C}^{n+1}[a,b]$*, then the local truncation error is of order* $\mathcal{O}(h^n)$.

Runge Kutta Methods

Theorem

Suppose that $f(t, y)$ *and all its partial derivatives of the order less than or equal to* $n+1$ *are continuous on* $D = \{(t, y) : a \leq t \leq b, c \leq y \leq d\}$ *and let* $(t_0, y_0) \in D$*. For every* $(t, y) \in D$ *there exist a* ξ *between* t *and* t_0 *and* μ *between* y *and* y_0 *with*

$$f(t, y) = P_n(t, y) + R_n(t, y),$$

where

$$\begin{aligned} P_n(t, y) \quad = \quad & f(t_0, y_0) + \left[(t - t_0)\frac{\partial f}{\partial t}(t_0, y_0) + (y - y_0)\frac{\partial f}{\partial y}(t_0, y_0)\right] \\ & + \left[\frac{(t - t_0)^2}{2}\frac{\partial^2 f}{\partial t^2}(t_0, y_0) + (t - t_0)(y - y_0)\frac{\partial^2 f}{\partial t \partial y}(t_0, y_0) + \frac{(y - y_0)^2}{2}\frac{\partial^2 f}{\partial y^2}(t_0, y_0)\right] \\ & + \cdots + \left[\frac{1}{n!}\sum_{j=0}^{n} {}^{n}C_j (t - t_0)^{n-j}(y - y_0)^j \frac{\partial^n f}{\partial t^{n-j}\partial y^j}(t_0, y_0)\right], \end{aligned}$$

and

$$R_n(t, y) = \frac{1}{(n+1)!}\sum_{j=0}^{n+1} {}^{n+1}C_j (t - t_0)^{n+1-j}(y - y_0)^j \frac{\partial^{n+1} f}{\partial t^{n+1-j}\partial y^j}(\xi, \mu).$$

Runge Kutta Methods

Taylor-method of order 2

$$T^{(2)}(t,y) = f(t,y) + \frac{h}{2} f'(t,y)$$

We express $f'(t,y)$ in terms of $a_1 f(t+\alpha_1, y+\beta_1)$ for some a_1, α_1, β_1 s.t. the error is of order 2.

$$f'(t,y) = \frac{d}{dt} \underset{\downarrow \atop f(t,\, y(t))}{f(t,y)} = \frac{\partial f}{\partial y} \boxed{\frac{dy}{dt}} + \frac{\partial f}{\partial t} = f(t,y)\frac{\partial f}{\partial y} + \frac{\partial f}{\partial t}$$

$$\Rightarrow T^{(2)}(t,y) = \left\{ f(t,y) + \frac{h}{2}\left[f(t,y)\frac{\partial f}{\partial y} + \frac{\partial f}{\partial t}\right]\right\}. \quad -①$$

Taylor expansion of $a_1 f(t+\alpha_1, y+\beta_1)$ —②

Neglect

$$a_1 f(t+\alpha_1, y+\beta_1) = a_1 \left[f(t,y) + \alpha_1 \frac{\partial f}{\partial t} + \beta_1 \frac{\partial f}{\partial y} + R_1(\cancel{\ ,}\ )\right]$$

Remainder term of 2^{nd}

Runge Kutta Methods

comparing ① and ②

$$a_1 = 1; \quad \alpha_1 = \frac{h}{2}; \quad \beta_1 = \frac{h}{2} f(t,y).$$

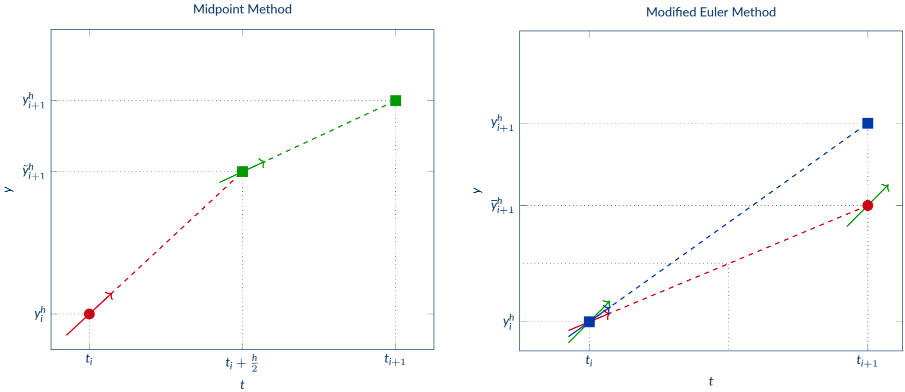
$$\Rightarrow T^{(2)}(t,y) = f\left(t+\frac{h}{2},\ y+\frac{h}{2}f(t,y)\right) - R(\qquad).$$

$$y_0 = \alpha$$

$$\underbrace{y_{i+1} = y_i + h f\left(t_i+\frac{h}{2},\ y_i+\frac{h}{2}f(t_i,y_i)\right) \text{ for } i = 0,\dots,n-1.}_{\text{Mid-point method.}}$$

Runge Kutta Methods

Runge Kutta Methods



Geometrical view of the Midpoint and Modified Euler methods.

Runge Kutta Methods

Ordinary Differential Equations

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Ordinary Differential Equations