Lecture 12: Differential Equations

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16^{th} April 2026

A m-step linear multistep method is

$$y'(t) = f(t, y)$$

is defined by

$$\boxed{y_{i+1} = \sum_{j=0}^{m-1} a_j\, y_{i+1-m+j} + h \sum_{j=0}^{m} b_j f(t_{i+1-m+j}, y_{i+1-m+j})}$$

for $i = m-1, m, \dots, n-1$, $m \geq 2$; $h = \frac{b-a}{n}$; $\{a_j\}$, $\{b_j\}$ are constants

$$y_0 = \alpha; \quad y_1 = \alpha_1, \dots, y_{m-1} = \alpha_{m-1}.$$

<u>$m=3$</u>

$$y_{i+1} = a_2 y_i + a_1 y_{i-1} + a_0 y_{i-2} + \underbrace{h \sum_{j=0}^{3} b_j f(t_{i-2+j}, y_{i-2+j})}.$$

$$h b_0 f(t_{i-2}, y_{i-2}) + h b_1 f(t_{i-1}, y_{i-1}) + h b_2 f(t_i, y_i) + \boxed{h b_3 f(t_{i+1}, y_{i+1})}$$

RK-methods : $y_{i+1} = \phi(t_i, y_i) \to$ Some function] Explicit methods.

$i+1 \to$ depends on $i, i-1, \ldots, 0$ -

$y_{i+1} = \phi_0(t_i, y_i, t_{i+1}, y_{i+1}) \to$ Implicit method

$i+1 \to$ depends on $i+1, i, \ldots, 0$

Adam- Bashforth

$y_0 = \alpha, \quad y_1 = \alpha_1, \quad y_2 = \alpha_2, \quad y_3 = \alpha_3$ — 4^{th} Order multistep $O(h^4)$

$$y_{i+1} = y_i + \frac{h}{24}\left[55 f(t_i, y_i) - 59 f(t_{i-1}, y_{i-1}) + 37 f(t_{i-2}, y_{i-2}) - 9 f(t_{i-3}, y_{i-3})\right]$$

— Explicit

Adam- Moulton

$y_0 = \alpha, \quad y_1 = \alpha_1, \quad y_2 = \alpha_2$ — 4^{th} Order $O(h^4)$

$$y_{i+1} = y_i + \frac{h}{24}\left[9 f(t_{i+1}, y_{i+1}) + 19 f(t_i, y_i) - 5 f(t_{i-1}, y_{i-1}) + f(t_{i-2}, y_{i-2})\right]$$

— Implicit

α_0: Initial Cond^n

$\alpha_1, \alpha_2, \ldots$: Explicit RK-method; for eg. α_1: y_1 = Euler method

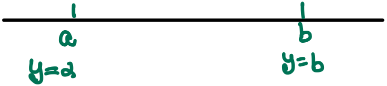
α_2: y_2 = Euler method.

IVP: $\dfrac{dy}{dt} = f(t, y)\,;\quad a \le t \le b$
$y(a) = \alpha$
$\Big\} \to$ Time-dependent equations

Boundary Value Problems (BVP):

$$y''(x) = f(x, y, y')\,; \qquad a \le x \le b$$

$$y(a) = \alpha\,; \qquad y(b) = \beta$$



$d \to$ Total
$\partial \to$ Partial

$$f(x, y) = y^2 + x$$

$\dfrac{\partial f}{\partial y} = 2y\,;\quad \dfrac{\partial f}{\partial x} = 1 \to$ y is not a function of x.

Theorem

Suppose that the function f in the boundary value problem

$$y''(x) = f(x, y(x), y'(x)), \quad a \leq x \leq b,$$

with boundary conditions $y(a) = \alpha$ *and* $y(b) = \beta$*, is continuous on the set*

$$D = \{(x, y, y') : a \leq x \leq b,\ -\infty < y < \infty,\ -\infty < y' < \infty\},$$

and that the partial derivatives f_y *and* $f_{y'}$ *are also continuous on D. If*

1. $f_y(x, y, y') > 0$ *for all* $(x, y, y') \in D$*, and*
2. *there exists a constant* $M > 0$ *such that*

$$|f_{y'}(x, y, y')| \leq M \quad \text{for all } (x, y, y') \in D,$$

then the boundary value problem has a unique solution.

$f(x, y, y') = p(x)y' + q(x)y + r(x)$

Corollary

Suppose the linear boundary value problem

$$y''(x) = p(x)y' + q(x)y + r(x), \quad a \leq x \leq b,$$

with boundary conditions $y(a) = \alpha$ and $y(b) = \beta$ satisfies:

1. *$p(x)$, $q(x)$, and $r(x)$ are continuous on $[a, b]$,*
2. *$q(x) > 0$ for all $x \in [a, b]$.*

Then the boundary value problem has a unique solution.

$a = x_0, x_1, x_2, \ldots, x_n, b = x_{n+1}$

$h = \frac{b-a}{n+1}$; $x_i = a + ih$ for $i = 0, \ldots, n+1$

$\boxed{y_i''} = p_i y_i' + q_i y_i + r_i \rightarrow y_i = y(x_i),\ p_i = p(x_i)$

compute, independent of other derivative terms, i.e., y_i'' depend on y_i ; h, x_i [i can be anything]

$i+1, i-1, i+2, i-2$

Use a Taylor series

$$y_{i+1} = y(x_i + h) = y_i + h y_i' + \frac{h^2}{2} y_i'' + O(h^3).$$

$$y_{i+1} = y_i + h y_i' + \frac{h^2}{2} y_i'' + \frac{h^3}{6} y_i^{(3)} + \frac{h^4}{4!} y_i^{(4)} + O(h^5) \quad (1)$$

$O(h)$ $\Big[\Rightarrow$

$$y_i'' = \underbrace{(y_{i+1} - y_i)\frac{2}{h^2} - (h y_i')\frac{2}{h^2}}_{\text{Substitute}} \underbrace{- \frac{h^3}{6} y_i^{(3)} \cdot \frac{2}{h^2}}_{\text{Order}} - O(h^2)$$

We can get better!

$$y_{i-1} = y_i - hy_i' + \frac{h^2}{2} y_i'' - \frac{h^3}{6} y_i^{(3)} + \frac{h^4}{4!} y_i^{(4)} + \mathcal{O}(h^5). \quad -(2)$$

Add (1) & (2) →

$$y_{i+1} + y_{i-1} = 2y_i + h^2 y_i'' + \frac{h^4}{12} y_i^{(4)} + \mathcal{O}(h^6)$$

$$\Rightarrow \quad y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + \underbrace{\frac{h^2}{12} y_i^{(4)}}_{\text{LTE} \to \mathcal{O}(h^2)} + \mathcal{O}(h^4)$$

Computation of y_i'

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2} y_i'' + \frac{h^3}{6} y_i^{(3)} + \frac{h^4}{4!} y_i^{(4)} + \mathcal{O}(h^5) \quad -(3)$$

$$y_i' = \frac{y_{i+1} - y_i}{h} + \mathcal{O}(h)$$

$$y_{i-1} = y_i - h y_i' + \frac{h^2}{2} y_i'' - \frac{h^3}{6} y_i^{(3)} + \frac{h^4}{4!} y_i^{(4)} + O(h^5). \quad (4)$$

Subtract (2) from (1)

$$y_{i+1} - y_{i-1} = 2h y_i' + \frac{h^3}{3} y_i^{(3)} + O(h^5)$$

$$\Rightarrow \boxed{y_i' = \frac{y_{i+1} - y_{i-1}}{2h}} + O(h^2)$$ - 2^{nd}-order term

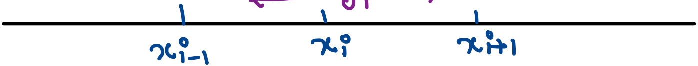
Central difference of 1^{st}-order derivative

$y_i' \longrightarrow$

$y_i'' \longrightarrow$

$\longleftarrow y_i \longrightarrow$

$y_i' = \frac{y_{i+1} - y_i}{h}$ (Forward difference)



$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$ → Central differencing for 2^{nd} order derivative

Using central differencing.

$$y_i'' = p_i y_i' + q_i y_i + r_i$$

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = p_i \left(\frac{y_{i+1} - y_{i-1}}{2h} \right) + q_i y_i + r_i \quad ; \quad i = 1, \dots, n$$

$$\left(\frac{1}{h^2} + \frac{p_i}{2h} \right) y_{i-1} + \left(-\frac{2}{h^2} + q_i \right) y_i + \left(\frac{1}{h^2} - \frac{p_i}{2h} \right) y_{i+1} = r_i$$

$i=1$; $$\left(\frac{1}{h^2} + \frac{p_1}{2h} \right) \underbrace{y_0}_{=\alpha} + \left(-\frac{2}{h^2} + q_1 \right) y_1 + \left(\frac{1}{h^2} - \frac{p_1}{2h} \right) y_2 = r_1$$

$$\left(-\frac{2}{h^2} + q_1 \right) y_1 + \left(\frac{1}{h^2} - \frac{p_1}{2h} \right) y_2 = r_1 - \left(\frac{1}{h^2} + \frac{p_1}{2h} \right) \alpha$$

$i=2$; $$\left(\frac{1}{h^2} + \frac{p_2}{2h} \right) y_1 + \left(-\frac{2}{h^2} + q_2 \right) y_2 + \left(\frac{1}{h^2} - \frac{p_2}{2h} \right) y_3 = r_2$$

$\vdots$

$i = n-1$

$i = n$

$$\left(\frac{1}{h^2}+\frac{p_n}{2h}\right)y_{n-1} + \left(\frac{-2}{h^2}+q_n\right)y_n + \left(\frac{1}{h^2}-\frac{p_n}{2h}\right)\underbrace{y_{n+1}}_{\beta} = r_n$$

$$\Rightarrow \left(\frac{1}{h^2}+\frac{p_n}{2h}\right)y_{n-1} + \left(\frac{-2}{h^2}+q_n\right)y_n = r_n - \left(\frac{1}{h^2}-\frac{p_n}{2h}\right)\beta$$

Collect;

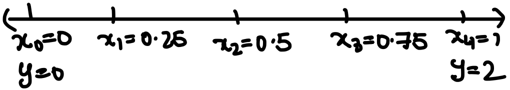
$$\underline{A}\,\underline{y} = \underline{b}$$

where $\underline{y} = [y_1 \;\; y_2 \; \dots \; y_n]$

$$A = \begin{bmatrix} \frac{-2}{h^2}+q_1 & \frac{1}{h^2}-\frac{p_1}{2h} & 0 & 0 & 0 & \dots & 0 \\ \frac{1}{h^2}+\frac{p_2}{2h} & \frac{-2}{h^2}+q_2 & \frac{1}{h^2}-\frac{p_3}{2h} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & & & \\ 0 & 0 & 0 & & & & \frac{-2}{h^2}+q_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$\underline{b} = \begin{bmatrix} x_1 - \left(\frac{1}{h^2} + \frac{p_1}{2h}\right) \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n - \left(\frac{1}{h^2} - \frac{p_n}{2h}\right) \end{bmatrix}$$

<u>Example</u>: $y'' = 4(y - x)$; $0 \le x \le 1$; $y(0) = 0$; $y(1) = 2$; $h = 0.25$



$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$ → Subs in equation.

$$\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) = 4(y_i - x_i)$$

$i=1$; $\frac{y_2 - 2y_1 + y_0}{h^2} = 4(y_1 - x_1) \Rightarrow \frac{y_2}{h^2} - \frac{2y_1}{h^2} - 4y_1 = -4x_1 - \frac{y_0}{h^2}$

$$\Rightarrow \left(-4 - \frac{2}{h^2}\right)y_1 + \frac{1}{h^2}y_2 = -1$$

$i=2$; $\frac{y_3 - 2y_2 + y_1}{h^2} = 4(y_2 - x_2) \Rightarrow \left(\frac{1}{h^2}\right)y_1 + \left(\frac{-2}{h^2} - 4\right) + \frac{1}{h^2}y_3 = -4 \times 0.5$
$= -2$

$i=3$; $\frac{y_4 - 2y_3 - y_2}{h^2} = 4(y_3 - x_3) \Rightarrow \left(-\frac{1}{h^2}\right)y_2 + \left(\frac{-2}{h^2} - 4\right)y_3 = -4 \times 0.75 - \frac{y_4}{h^2}$
$= -3 - \frac{2}{h^2}$

$$\begin{bmatrix} -\left(\frac{2}{h^2}+4\right) & \frac{1}{h^2} & 0 \\ \frac{1}{h^2} & -\left(\frac{2}{h^2}+4\right) & \frac{1}{h^2} \\ 0 & \frac{1}{h^2} & -\left(\frac{2}{h^2}+4\right) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ -3-\frac{2}{h^2} \end{bmatrix}$$

Drawback

1. Computation of new values require re-gridding (re-meshing) or interpolation.