Lecture 13: Differential Equations

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Outline

A or B should be diagonal. then $\{\lambda_i + \beta_i\}$ holds ✗

$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$; $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\downarrow$ Diagonal $\{1, 2\}$

$\downarrow$ $\{1, -1\} \rightarrow$ A + B $\{3, 1\}$; $\{0, 3\}$ $\not\equiv$

$A+B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$; $\frac{3 \pm \sqrt{5}}{2}$; $\alpha A, B$ should be αI.

Shifted power method $\to$ 1 . 1B

3 $\to$ $\int_0^1 e^x \, dx$; Abs error

$\downarrow$ $e^x|_0^1 = \underline{|e - 1 - \tilde{x}|}$ $\rightarrow$ Num. value

2: $\alpha = ?$

x^2 in $p(x)$

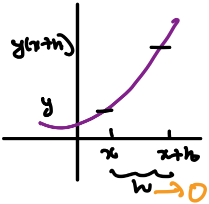
PDE is a multivariable function with one or more of its partial derivatives.

$u \to$ Unknown;

$u : \Omega \to \mathbb{R}$; $\Omega \subseteq \mathbb{R}^d$; $d \geq 2$

$d=2$; $\mathbb{R}^2$, $(x,y) \to u(x,y)$

$d=3$; $\mathbb{R}^3$, $(x,y,z) \to u(x,y,z)$



Partial Derivatives: Derivative: $\frac{dy}{dx} = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h}$

P. Derivative: $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ $\left(\frac{\partial u}{\partial z}\right)$

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(x+h, y) - u(x,y)}{h}, \quad \frac{\partial u}{\partial y} = \lim_{k \to 0} \frac{u(x, y+k) - u(x,y)}{k}$$

Gradient $= \nabla u := \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$; Divergence, Curl (vectors, not scalars).

Higher-order P.D.: $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \underbrace{\frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y \partial x}}_{\text{Different : Same if } u \text{ is continuous.}}$; $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$

$u = x^2 + xy + y^2$

$\frac{\partial u}{\partial x} = 2x + y$; $\frac{\partial u}{\partial y} = x + 2y$; $\frac{\partial^2 u}{\partial x \partial y} = 1$; $\frac{\partial^2 u}{\partial y \partial x} = 1$. $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = 2$

$\frac{\partial^2 u}{\partial y^2} = 2$.

Linear, 2^{nd}-order-PDE in 2D

$$A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u = G,$$

where $A, B, C, \ldots G$ are constants: $u_{xx} = \frac{\partial^2 u}{\partial x^2}$

Classification: Discriminant: $B^2 - 4AC$

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = G$$

1. $B^2-4AC<0$: Elliptic PDE
2. $B^2-4AC=0$: Parabolic PDE
3. $B^2-4AC>0$: Hyperbolic PDE

$\rightarrow$ course

$$\frac{\partial u}{\partial t} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = f$$

Heat equation in 2D

$u(t,x,y)$

1. Laplace equation / Poisson equation

$$-\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = f \; ; \quad B=0;\ A=C=-1$$

$$B^2-4AC = -4 < 0$$

2. Heat equation:

$u(t, x)$: t — Time, x — Space

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f \; ; \quad B=0;\ C=0;\ A=-1$$

$$B^2-4AC=0$$

3. Wave equation

$$u(t,x): \quad \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = f \; ; \quad B=0;\ A=1,\ C=-1$$

$$B^2-4AC = 4 > 0$$

Laplace / Poisson equation

$$-\Delta u = f \quad \text{in } \Omega \subseteq \mathbb{R}^2 \quad ; \quad \Delta = \nabla \cdot \nabla$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$$

where
$$\Delta = \sum_{i=1}^{2} \frac{\partial^2 u}{\partial x_i} = \frac{\partial^2 u}{\partial x_1} + \frac{\partial^2 u}{\partial x_2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$u = g$ on $\partial\Omega \to$ Boundary.

↳ Dirichlet BC.

$f \neq 0 \to$ Poisson equation

$f = 0 \to$ Laplace equation.

$\partial_n u = g$ on $\partial\Omega$ ✗

↳ Neumann BC

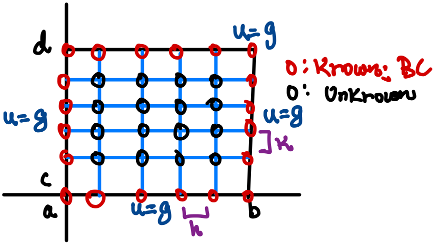
$\partial_n u = \nabla u \cdot \vec{n}$ ✗

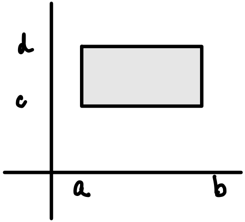
$\to$ Mix : Robin BC ✗

1. Electrical / Material: Electric potential inside capacitor
2. Mechanical: Model incompressible flow.
3. Civil : Ground water flow
4. Chemical: Poisson-Boltzmann electrostatics / Molecular dynamics.

Case: Ω: Rectangular domain, $(a,b)\times(c,d) \subseteq \mathbb{R}^2$.

Boundary condition: Dirichlet





Same way as BVP for ODE.

1. Grid or mesh the domain:

$$h = \frac{b-a}{M} \; ; \quad k = \frac{d-c}{M} \; ;$$ we have M points in both direction

2. $x_i = a + ih$; $y_j = c + jk$; $i, j = 0, 1, \ldots, M$

$x_0 = a$ $\quad y_0 = c$

$x_M = b$ $\quad y_M = d$

We need to solve: $-\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = f(x,y)$ in $\Omega \subseteq \mathbb{R}^2$

$u(x,y) = g(x,y)$ on $\partial\Omega$

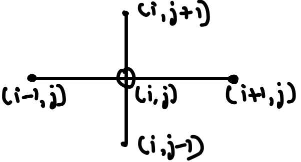
$$u(x+h, y) = u(x,y) + h u_x(x,y) + \frac{h^2}{2} u_{xx}(x,y) + \frac{h^3}{3!} u_{xxx}(x,y) + O(h^4)$$

$$u(x-h, y) = u(x,y) - h u_x(x,y) + \frac{h^2}{2} u_{xx}(x,y) - \frac{h^3}{3!} u_{xxx}(x,y) + O(h^4)$$

$$u_{xx} = \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2} + O(h^2)$$

Similarly: $u_{yy} = \dfrac{u(x, y+k) - 2u(x,y) + u(x,y-k)}{k^2} + O(k^2)$

We compute the solution at $(i,j) \to$ Solve Poisson equation at (i,j)



at $(i,j) \to u(x_i, y_j)$

$(i\pm1, j) \Rightarrow u(x_{i\pm1}, y_j) = u(x_i \pm h, y_j)$

$(i, j\pm1) \to u(x_i, y_{j\pm1}) = u(x_i, y_j \pm k)$

$$-\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\bigg|_{(i,j)} = f(x,y)\big|_{(i,j)}$$

$$u_{xx}\big|_{(i,j)} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + O(h^2)$$

$$u_{yy}\big|_{(i,j)} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} + O(k^2)$$

Substitute : $$-\left[\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j-1}}{k^2}\right] = f_{i,j} + O(h^2+k^2)$$

Drop $O(h^2+k^2)$ to get numerical solution $\{\underline{u}_{i,j}\}$

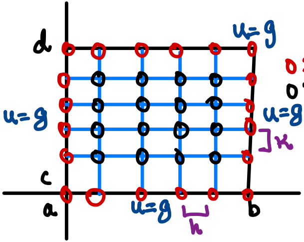
$$-\left[\frac{\underline{u}_{i+1,j}-2\underline{u}_{i,j}+\underline{u}_{i-1,j}}{h^2}+\frac{\underline{u}_{i,j-1}-2\underline{u}_{i,j}+\underline{u}_{i,j-1}}{k^2}\right]=\underline{f}_{i,j}$$

$$\underbrace{\Rightarrow\ 2\left(\frac{1}{h^2}+\frac{1}{k^2}\right)\underline{u}_{i,j}-\left[\frac{1}{h^2}\left(\underline{u}_{i+1,j}+\underline{u}_{i-1,j}\right)+\frac{1}{k^2}\left(\underline{u}_{i,j+1}+\underline{u}_{i,j-1}\right)\right]=\underline{f}_{i,j}\qquad i,j=1,\dots,M-1}_{\text{Five-point stencil}}$$

At $i=1$, $j=1$, $i=M-1$, $j=M-1$

$i=1$: $$2\left(\frac{1}{h^2}+\frac{1}{k^2}\right)\underline{u}_{1,j}-\left[\frac{1}{h^2}\left(\underline{u}_{2,j}+\underbrace{\underline{u}_{0,j}}_{\text{Known = at } x=a}\right)+\frac{1}{k^2}\left(\underline{u}_{1,j+1}+\underline{u}_{1,j-1}\right)\right]=\underline{f}_{1,j}$$

Elliptic PDEs



o : Known: BC

o : Unknown

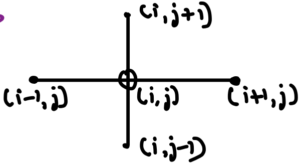
System of equations $A\underline{u} = \underline{b}$

$A \in \mathbb{R}^{(M-1)^2 \times (M-1)^2}$; $\underline{u}, \underline{b} \in \mathbb{R}^{(M-1)^2}$

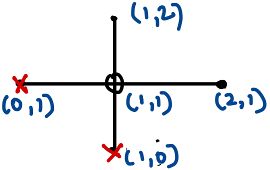
Specific case : $f \equiv 0 \to$ Laplace equation; $h = k$

$$\frac{4}{h^2} \underline{u}_{i,j} - \frac{1}{h^2}\left[\underline{u}_{i\pm1,j} + \underline{u}_{i,j\pm1}\right] = 0$$

$$\Rightarrow \underline{u}_{i,j} = \frac{1}{4}\left[\underline{u}_{i\pm1,j} + u_{i,j\pm1}\right]$$



$\to$ Mean value Property of Harmonic



$$(a,b)\times(c,d) = (0,1)\times(0,1)$$

$$A\underline{u} = \underline{b}$$

$$\underline{u} = [\underline{u}_1 \quad \underline{u}_2 \quad \cdots \quad \underline{u}_{m-1}]^T$$

$$\underline{u}_i = [\underline{u}_{i,1} \quad \underline{u}_{i,2} \quad \underline{u}_{i,3} \quad \cdots \quad \underline{u}_{i,m-1}]^T$$

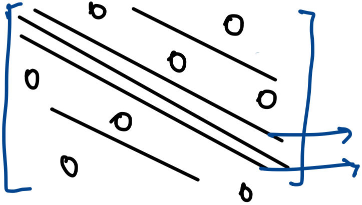
$$(i,j) = (1,1) \rightarrow h = k \Rightarrow h$$

$$\frac{4}{h^2}\underline{u}_{1,1} - \frac{1}{h^2}\Big(\underline{u}_{2,1} + \underbrace{\underline{u}_{0,1}}_{\underline{g}_{0,1}} + \underline{u}_{1,2} + \underbrace{\underline{u}_{1,0}}_{\underline{g}_{1,0}}\Big) = \underline{f}_{1,1}$$

$$\Rightarrow \quad 4\underline{u}_{1,1} - (\underline{u}_{2,1} + \underline{u}_{1,2}) = \underline{f}_{1,1}h^2 + \underline{g}_{0,1} + \underline{g}_{1,0}$$

Elliptic PDEs

A → Pentadiagonal system → 5 diagonals



$A \in \mathbb{R}^{(M-1)^2 \times (M-1)^2}$

Non-zero entries $\sim (M-1)^2$

A → Sparse matrix → Large fraction of zero entries.

Iterative solvers are attractive! as matrix is sparse.

Finite Difference Methods for elliptic PDE:

→ Convergence, Consistency, Stability