# Lecture 14: Differential Equations

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# Outline

We consider the one-dimensional heat equation ; $u(x,t)$

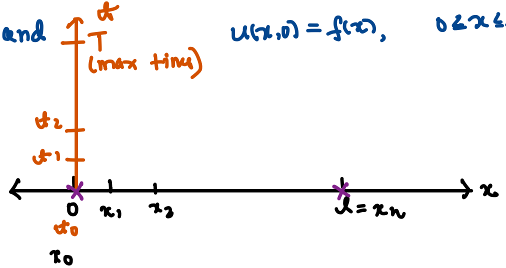
$$u_t = u_{xx} \quad , 0 < x < l, \quad t > 0$$

Subject to, $u(0,t) = u(l,t) = 0 \ , t > 0$

and $u(x,0) = f(x), \quad 0 \le x \le l$

; $f(0) = f(l) = 0$

Boundary initial compatibility condition



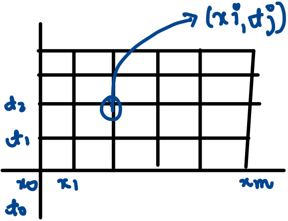
Discretisation follows the same as elliptic PDE.

We discretise in space with m-points, and

$$h = \frac{l}{m}$$

and

$$k = \frac{T}{n}, \quad n\text{-points in time.}$$




We approximate the solution at $(x_i, t_j)$.

$$\left.\frac{\partial u}{\partial t}\right|_{(x_i, t_j)} = \left.\frac{\partial^2 u}{\partial x^2}\right|_{(x_i, t_j)}$$

For $2^{nd}$-order $\frac{\partial^2 u}{\partial x^2}$, we know

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{(x_i, t_j)} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + O(h^2) \quad \text{—} (1)$$

For time;

$$\frac{\partial u}{\partial t} = \lim_{\Delta t \to 0} \frac{u(x, t+\Delta t) - u(x,t)}{\Delta t}$$

} Forward difference
Backward difference

Forward difference;

$$\left.\frac{\partial u}{\partial t}\right|_{(x_i, t_j)} = \frac{u_{i,j+1} - u_{i,j}}{k} + O(k) \quad (2)$$

} Derivation

$$u_{i,j+1} = u(x_i, t_{j+1}) = u(x_i, t_j + k) = u(x_i, t_j) + k \left.\frac{\partial u}{\partial t}\right|_{(x_i,t_j)} + k^2 \left.\frac{\partial^2 u}{\partial t^2}\right|_{(x_i,t_j)} + O(k^3)$$

Divide

Substitute (1) and (2) in $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + O(h^2 + k)$$

Dropping the $\mathcal{O}(\ )$ terms we get

$$\frac{u_{i,j+1} - u_{i,j}}{K} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

$j=0 \rightarrow u_{i,0} \rightarrow$ Initial condition $= f_i \ (= f(x_i))$

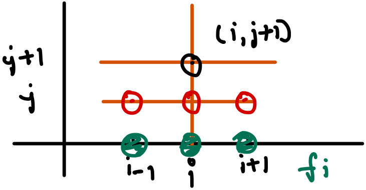
$$\left.\frac{u_{i,1} - u_{i,0}}{K} = \frac{u_{i+1,0} - 2u_{i,0} + u_{i-1,0}}{h^2}\right\} \Rightarrow u_{i,1} = \frac{K}{h^2}\left(u_{i+1,0} - 2u_{i,0} + u_{i-1,0}\right) + u_{i,0}$$

$$\Rightarrow u_{i,1} = \frac{K}{h^2} u_{i+1,0} + \left(1 - \frac{2K}{h^2}\right) u_{i,0} + \frac{K}{h^2} u_{i-1,0}$$

For any $j$

$$\left.\Rightarrow u_{i,j+1} = \frac{K}{h^2} u_{i+1,j} + \left(1 - \frac{2K}{h^2}\right) u_{i,j} + \frac{K}{h^2} u_{i-1,j}.\right]$$ Explicit method

## Forward Euler method




$$\underline{u}^{(j+1)} = A^{FE}\,\underline{u}^{(j)}$$

BCs

$$A^{FE} = \begin{bmatrix} 1-2\lambda & \lambda & 0 & \cdots & 0 \\ \lambda & 1-2\lambda & \lambda & \cdots & 0 \\ 0 & \lambda & 1-2\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1-2\lambda \end{bmatrix}, \quad \underline{u}^{(j)} = \begin{bmatrix} \underline{u}_{1,j} \\ \underline{u}_{2,j} \\ \vdots \\ \underline{u}_{m-1,j} \end{bmatrix}, \quad \lambda = \frac{K}{h^2}$$

BCs

**Example:** $f(x) = \sin(\pi x)$, $T = 1$, $l = 1$; $u(x,t) = \exp(-\pi^2 t)\sin(\pi x)$

|  | $h = 0.1,\ k = 0.01$ | $h = 0.1,\ k = 0.0005$ |
|---|---|---|
| $x_i$ | $\|u(x_i, 0.5) - u^{FE}(x_i, 0.5)\|$ | $\|u(x_i, 0.5) - u^{FE}(x_i, 0.5)\|$ |
| 0.10 | 1.578e+06 | 1.219e-04 |
| 0.20 | 2.999e+06 | 1.678e-04 |
| 0.30 | 4.122e+06 | 1.973e-04 |
| 0.40 | 4.836e+06 | 2.075e-04 |
| 0.50 | 5.074e+06 | 1.973e-04 |
| 0.60 | 4.816e+06 | 2.079e-04 |
| 0.70 | 4.089e+06 | 1.678e-04 |
| 0.80 | 2.966e+06 | 1.219e-04 |
| 0.90 | 1.558e+06 | 6.411e-05 |

**Table 1:** Pointwise error at $t = 0.5$ for the Forward Euler method with $h = 0.1$ and different values of $k$.

Stability: Let $\underline{e}^{(0)} = [\underline{e}_1^{(0)} \; \underline{e}_2^{(0)} \; \ldots \; \underline{e}_{m-1}^{(0)}]^T$ be the error in the I.C.

$\underline{u}^{(1)} = A^{FE} \underline{u}^{(0)} \rightarrow$ Solution you want

Reality $\Rightarrow$ $\underline{u}^{(1)} = A^{FE}(\underline{u}^{(0)} + \underline{e}^{(0)})$

at $j=2$; $\underline{u}^{(2)} = A^{FE} \underline{u}^{(1)} = (A^{FE})^2 \underline{u}^{(0)} + (A^{FE})^2 \underline{e}^{(0)}$

$\vdots$

$j=n$; $\underline{u}^{(n)} = (A^{FE})^n \underline{u}^{(0)} + \underbrace{(A^{FE})^n \underline{e}^{(0)}}_{\text{Error propagation.}}$

From Linear algebra

$$\|(A^{FE})^n \underline{e}^{(0)}\| \leq \underbrace{\|(A^{FE})^n\|}_{\leq C \rightarrow \text{ It's fine}} \cdot \|\underline{e}^{(0)}\|$$

If $c<1$, then we are good, as $c^n$ is small.

Linear algebra: $\rho(A) \le \|A\| \to \rho(A) =$ Spectral radius

$= \max\{|\beta|,\ \beta$ is e-value of $A\}$.

If $\rho(A^n) \le 1$; then we are done.

$\rho(A)$ gives the result as $\rho(A)$ and $\rho(A^n)$ are related

$\rho(A) \le 1$ then we are done

Eigenvalue of $A^{FE}$

$$\beta_i = 1 - 4\lambda\left(\sin\left(\frac{i\pi}{2m}\right)\right)^2, \qquad i = 1, 2, \ldots, m-1$$

$$\rho(A^{FE}) = \max_{1 \le i \le m-1} |\beta_i|$$

$$\max_{1\le i\le m-1} |\beta_i| \le \left|1 - 4\lambda\left(\sin\left(\frac{i\pi}{2m}\right)\right)^2\right| \le 1$$

as $\sin^2(\ ) \le 1$

$\Rightarrow \quad |1-4\lambda| \le 1 \quad ; \quad -1 \le 1-4\lambda \le 1 \Rightarrow 0 \le 4\lambda \le 2$

$\Rightarrow \quad \underbrace{0 \le \lambda}_{} \le \frac{1}{2}$

Trivially true; $\lambda = \frac{k}{h^2}$

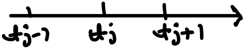
Stability $\rightarrow$ $\boxed{\frac{k}{h^2} \le \frac{1}{2}}$ $\rightarrow$ CFL condition.

Courant-Freidrich-Lax

Forward Euler: 1. Explicit $\rightarrow$ Fast

2. Stability: $\frac{k}{h^2} \le \frac{1}{2}$ $\rightarrow$ Conditionally stable

## Backward Euler




Everything remains same;

$$\left.\frac{\partial u}{\partial t}\right|_{(x_i,t_j)} = \frac{u_{i,j} - u_{i,j-1}}{k} + O(k)$$

Substitute this $\left.\frac{\partial u}{\partial t}\right|_{(x_i,t_j)} = \left.\frac{\partial^2 u}{\partial x^2}\right|_{(x_i,t_j)}$

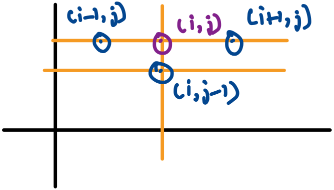
$$\frac{u_{i,j} - u_{i,j-1}}{k} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + O(h^2+k)$$

Dropping $O(\ )$ notation

$$\frac{\underline{u}_{i,j} - \underline{u}_{i,j-1}}{k} = \frac{\underline{u}_{i+1,j} - 2\underline{u}_{i,j} + \underline{u}_{i-1,j}}{h^2}$$

$$\Rightarrow \ -\lambda \underline{u}_{i-1,j} + (1+2\lambda)\underline{u}_{i,j} - \lambda \underline{u}_{i+1,j} = \underline{u}_{i,j-1}, \quad \lambda = \frac{k}{h^2}$$

$i = 1, \dots, m-1, \quad j = 1, \dots, n.$




→ Implicit method

at each $j$ → Solve a system of equation

Unconditionally stable!

Matrix formulation

$$A^{BE}\underline{u}^{(j)} = \underline{u}^{(j-1)}$$

$$\Rightarrow \underline{u}^{(j)} = \underbrace{\left(A^{BE}\right)^{-1}}_{C}\underline{u}^{(j-1)},$$ where

$$A^{BE} = \begin{bmatrix} 1+2\lambda & -\lambda & 0 & - & \cdots & 0 \\ -\lambda & 1+2\lambda & -\lambda & - & - & 0 \\ 0 & -\lambda & 1+2\lambda & - & - & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & - & 1+2\lambda \end{bmatrix}$$

Stability relies of $\rho((A^{BE})^{-1}) \leq 1$

Eigenvalue of $A^{BE}$: $\mu_i = 1 + 4\lambda\left(\sin\left(\frac{i\pi}{2m}\right)\right)^2$, $i = 1, \ldots, m-1$

$\geq 1$

$\Rightarrow$ spectral $\rho(A^{BE}) \geq 1$

If $\beta$ is e-value of $A$; $\beta^{-1}$ is e-value of $A^{-1}$

$\Rightarrow$ All e-value of $(A^{BE})^{-1} \leq 1$

$\Rightarrow \rho((A^{BE})^{-1}) \leq 1$

$\Rightarrow$ Method is unconditionally stable.

Backward Euler: 1. Implicit $\rightarrow$ Time consuming

2. Unconditional stability.

Drawback

1. $O(k)$ in time $\to$ FE & BE

   $\to$ To get $O(k^2)$ $\to$ averaging $\to$ Crank-Nicolson scheme } Conditional stable

   (Tutorial-5, Prob 3)

   Fluid flows (Finite volume)

2. Theta-scheme: $\theta \in [0,1]$ ; $\theta = \frac{1}{2} \to$ C.N.

   $\theta = 0 \to$ FE; $\theta = 1 \to$ BE

THE END!

| | $h = 0.1,\ k = 0.01$ | $h = 0.1,\ k = 0.0005$ |
|---|---|---|
| $x_i$ | $\lvert u(x_i, 0.5) - u^{\mathrm{BE}}(x_i, 0.5)\rvert$ | $\lvert u(x_i, 0.5) - u^{\mathrm{BE}}(x_i, 0.5)\rvert$ |
| 0.10 | 6.756e-04 | 1.195e-04 |
| 0.20 | 1.285e-03 | 2.274e-04 |
| 0.30 | 1.769e-03 | 3.130e-04 |
| 0.40 | 2.079e-03 | 3.679e-04 |
| 0.50 | 2.186e-03 | 3.868e-04 |
| 0.60 | 2.079e-03 | 3.679e-04 |
| 0.70 | 1.769e-03 | 3.130e-04 |
| 0.80 | 1.285e-03 | 2.274e-04 |
| 0.90 | 6.756e-04 | 1.195e-04 |

**Table 2:** Pointwise error at $t = 0.5$ for the Backward Euler method with $h = 0.1$ and different values of $k$.