Lecture 2: Root-Finding Methods

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Outline

(G8) (G7)
7/205 → 7/204

7/201 → 7/105
(G6) (G5)

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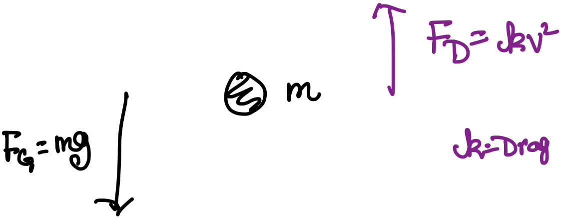


k: Drag coeffecient

For equillibirium, we need the net force to be zero and that v is called the terminal velocity

$F_G = F_D \Rightarrow mg = kv^2$

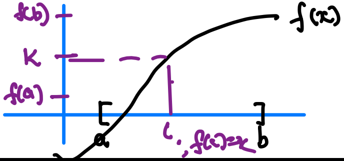
$v = ? \Rightarrow v = \sqrt{\frac{mg}{k}}$; $k \to$ Non-linear $\therefore k(v)$

$mg = k(v)\, v^2$

$ax^2 + bx + c = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Cubic, bi-quadratic
(3) (4).



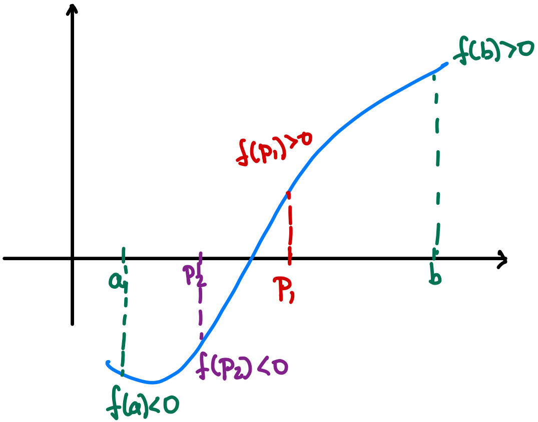
Theorem (Intermediate Value Theorem)

If $f \in \mathcal{C}[a,b]$ and $k \in [f(a), f(b)]$, then there exists a $c \in [a,b]$ such that

$$f(c) = k.$$

what happens if $K=0$? i.e. $f(c)=0$

1. Given the interval $[a,b]$: $f(a)\cdot f(b) < 0$ ← Assumption



1. $a_1 = a$; $b_1 = b$

   $p_1 = \frac{a_1+b_1}{2}$

2. $f(p_1) > 0$ or $f(p_1) < 0$

   $a_2 = a_1$; $b_2 = p_1$

3. $p_2 = \frac{a_2+b_2}{2}$

Repeat

$f(p_n) = 0 \} \to |f(p_n)| < tol$

$tol = 10^{-7}$; or max-iter

Bisection Method

Example: $f(x) = \sqrt{x} - \cos(x)$; $a = 0$; $b = 1$

$f(0) = -1$; <0 $f(1) \approx 0.4596 > 0$

Iteration 1: $p_1 = \frac{a_1 + b_1}{2} = 0.5$

$f(p_1) \approx -0.170475 < 0$

$a_2 = 0.5$ $b_2 = 1$

Iteration 2: $p_2 = \frac{a_2 + b_2}{2} = 0.75$

$f(p_2) \approx 0.13433 > 0$

$a_3 = 0.5$; $b_3 = 0.75$

Theorem

Suppose that $f \in C[a,b]$ and

$$f(a)f(b) < 0.$$

Then the bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ that converges to a zero p of f and satisfies

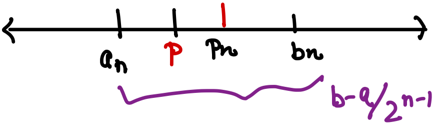
$$|p_n - p| \leq \frac{b-a}{2^n}, \quad n \geq 1.$$

Proof: $a_1 = a$; $b_1 = b$; $b_1 - a_1 = b - a$ $\qquad b-a \quad i=1$

$p_1 = \dfrac{a+b}{2}$; $\quad \underbrace{[a_1, p]}$ or $[p, b_1]$

$\left[a ; \dfrac{a+b}{2}\right]$ or $\left[\dfrac{a+b}{2}, b\right]$ $\qquad \dfrac{b-a}{2} \quad i=2$

$b_2 - a_2 = \dfrac{b-a}{2}$



$i=3 \quad b_3 - a_3 = \dfrac{b-a}{2^2}$

$\vdots$

$b_n - a_n = \dfrac{b-a}{2^{n-1}} \rightarrow$ In this interval $p_n = \dfrac{a_n+b_n}{2}$

$|p_n - p| \leq \dfrac{b_n - a_n}{2} \leq \dfrac{b-a}{2^n}$

$\boxtimes$

$$|P_n - P| \le \frac{b-a}{2^n}$$

$f(x) = x^3 + 4x^2 - 10$; $[1,2]$; $tol = 10^{-3}$

$$|P_n - P| \le \frac{2-1}{2^n} \le 10^{-3}$$

$$= \frac{1}{2^n} \le 10^{-3} \Rightarrow 10^3 \le 2^n$$

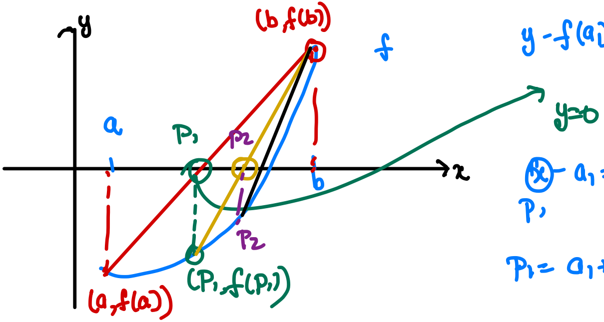
$$\Rightarrow 3 \le n \log_{10} 2$$

$$\Rightarrow n \geq 9.96$$

$$\simeq 10.$$

We start with same assumption as Bisection method, i.e,

$$f(a)\,f(b)<0$$



$$y - f(a_1) = \frac{(f(b_1)-f(a_1))}{b_1-a_1}(x-a_1)$$

$$x - a_1 = \frac{(a_1-b_1)f(a_1)}{f(b_1)-f(a_1)}$$

P_1

$$P_1 = a_1 + \frac{(a_1-b_1)\,f(a_1)}{f(b_1)-f(a_1)}$$

Based on sign of $f(P_1)$ you join the line and choose the new interval: if $f(a)<0$, $f(b)>0$

$f(P_1)<0 \rightarrow [a_2,b_2] = [P_1, b]$ or >0; $[a_2, P_1]$

Example: $f(x) = \sqrt{x} - \cos(x)$; $[0,1]$

$f(0) < 0$; $f(1) > 0$; ; $f(0) = -1$; $f(1) = 0.459697$

Iteration 1:

$$p_1 = a_1 + \frac{(a_1 - b_1) f(a_1)}{f(b_1) - f(a_1)}$$

$$= 0 + \frac{(0-1)(-1)}{0.459697 + 1} \approx 0.68507$$

$f(p_1) = 0.0533 > 0$

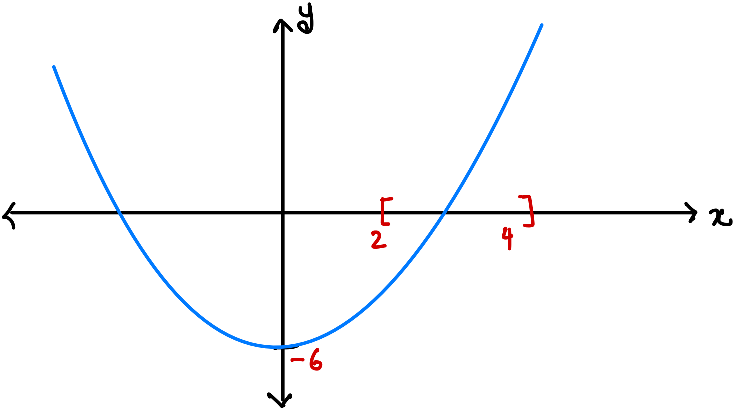
Iteration 2: $a_2 = 0$; $b_2 = 0.68507$

$\downarrow$

$p_2 = 0.6503$; $f(p_2) = 0.0101$

Regular-Falsi Method

$f(x) = x^2 - 6$; $[2,4]$



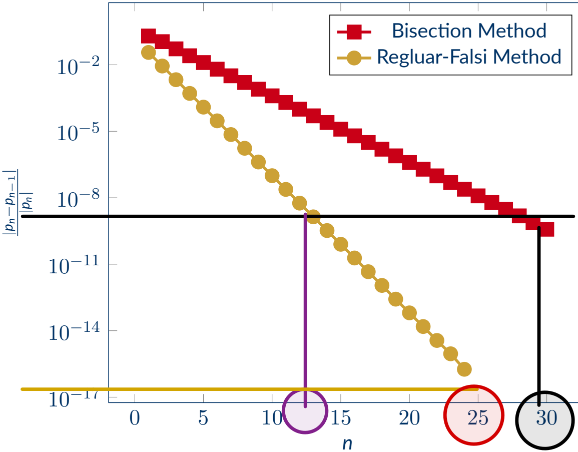


Figure 1: Relative error of the bisection method and the regular falsi method applied to $f(x) = x^2 - 6$ on the interval $[2, 4]$.

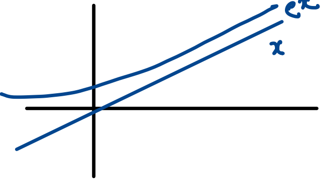
Definition: A number p is called a fixed point of g if
$$g(p)=p$$

examples: 1. $g(x)=\sin(x)$ has one FP, $p=0$

2. $g(x)=e^{x}$ has no F.P

3. $g(x)=x^2-6$ has 2 F.P, $3,-2$; $g(x)=x$

$\Rightarrow x^2-6=x \Rightarrow x=-2,3$



$\boxed{\text{Why F.P.s?}}$

$\boxed{R\cdot F}$: Find p s.t. $f(p)=0$

$\boxed{FP}$: Find p s.t. $g(p)=p$

$\Longrightarrow$ Given a R.F. problem we can construct a F.P. problem

$g(x) = x - f(x)$ or $x - \alpha f(x)$, $\alpha \neq 0$

$\alpha = h(x)$. $h(p) \neq 0$

$\Leftarrow$ Given a F.P. problem we can construct a RF problem

$f(x) = x - g(x)$

Theorem: Existence & Uniqueness of F.P.

1. If $g \in C[a,b]$ and $g(x) \in [a,b]$ $\forall x \in [a,b]$

$\in$: Belongs , $\forall$ = For all $\quad a \le g(x) \le b$

then g has a F.P in $[a,b]$.

2. If $g'(x)$ exist in $[a,b]$ and there exist $0<K<1$ s.t.

$$|g'(x)| \le K \quad \forall x \in (a,b).$$

then g has unique F.P.

Generate F.P.

1. Given a function $g(x)$ and initial approximation p_0; we generate $\{p_n\}_{n=0}^{\infty}$ with

$$p_1 = g(p_0)$$

$$p_2 = g(p_1)$$

$$p_3 = g(p_2)$$

$$\vdots$$

$$p_n = g(p_{n-1})$$

If $p_n \to p$ as $n \to \infty$ and g is continuous then

$$p := \lim_{n\to\infty} p_n = \lim_{n\to\infty} g(p_{n-1})$$

$$= g\left(\lim_{n\to\infty} p_{n-1}\right) = g(p) \Rightarrow p \text{ is a F.P.}$$