Lecture 3: Root-Finding Methods

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Theorem (Existence and Uniqueness of Fixed Points)

1. *If $g \in \mathcal{C}[a,b]$ and*

$$g(x) \in [a,b] \quad \forall x \in [a,b],$$

then g has at least one fixed point in $[a,b]$.

2. *If, in addition, $g'(x)$ exists on (a,b) and there exists a constant $0 < k < 1$ such that*

$$|g'(x)| \leq k \quad \forall x \in (a,b),$$

then g has exactly one fixed point in $[a,b]$.

e^x; e^x from $[a,b] \not\to [a,b]$; $[0,1]$; $e^0 = 1$, $e^1 = e$; $[0,1] \not\to [0,1]$

$x^2 - 6$; Existence $=$; $2x$; $[-4,4]$ it is not bdd by 1; hence 2 FPs

$p_0 = \alpha;$ $\quad p_1 = g(p_0)$

$p_2 = g(p_1)$

$\vdots$

$p_n = g(p_{n-1})$

If $p_n \to p$ as $n \to \infty$

then $p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} g(p_{n-1})$

$= g\left(\lim_{n\to\infty} p_{n-1}\right) = g(p)$

Example: $f(x) = x^3 + 4x^2 - 10$

$$g_1(x) = x - \frac{(x^3 + 4x^2 - 10)}{3x^2 + 8x}$$

$p_0 = 1.5;$ $\quad p_1 = g_1(p_0) \approx 1.373333;$ $\quad$ Rel. error $\frac{|p_1 - p_0|}{|p_1|} = 0.082444$

$p_2 = g_1(p_1) \approx 1.36526201;$ $\quad$ R. error $= 0.00591.$

$g_2(x) = x - x^3 - 4x^2 + 10;$ $\quad g_3(x) = \frac{1}{2}\sqrt{10 - x^3}.$

n	$p_n(g_1)$	$p_n(g_2)$	$p_n(g_3)$
0	1.5	1.5	1.5
1	1.3733333333333333	-0.875	1.286953767623375
2	1.3652620148746266	6.732421875	1.402540803539578
3	1.3652300139161466	-469.72001200169325	1.345458374023294
4	1.3652300134140969	102754555.18738511	1.375170252816038
5	1.3652300134140969	$-1.0849338705317464 \times 10^{24}$	1.360094192761733
6	–	$1.27705559144437 \times 10^{72}$	1.367846967592133
7	–	–	1.363887003884021
8	–	–	1.365916733390040
9	–	–	1.364878217193677
10	–	–	1.365410061169957

Table 1: Comparison of fixed-point iterations for g_1, g_2, and g_3.

Theorem

Let $g \in \mathcal{C}[a,b]$ be such that

$$g(x) \in [a,b] \quad \forall\, x \in [a,b].$$

Suppose, in addition, that $g'(x)$ exists on (a,b) and there exists a constant $0 < k < 1$ such that

$$|g'(x)| \leq k \quad \forall\, x \in (a,b).$$

Then, for any initial guess $p_0 \in [a,b]$, the sequence defined by

$$p_n = g(p_{n-1}), \qquad n \geq 1,$$

converges to the unique fixed point $p \in [a,b]$.

Corollary

If $g(x)$ satisfies the hypotheses of Theorem 2, then the error in using p_n to approximate p satisfies

$$|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\},$$

and

$$|p_n - p| \leq \frac{k^n}{1-k} |p_1 - p_0|, \qquad n \geq 1.$$

Consider $g_2(x) = x - x^3 - 4x^2 + 10$ and the interval $[1,2]$

Now, $g_2(1) = 6$, $g_2(2) = -12$.

So, $g_2 : [1,2] \not\to [1,2]$. Moreover: $g_2'(x) = 1 - 3x^2 - 8x$

and hence, $|g_2'(x)| > 1 \ \forall x \in [1,2]$.

$\leftrightarrow$ $g_3(x) = \frac{1}{2}\sqrt{10 - x^3}$; we compute $g_3'(x) = \frac{-3}{4} \frac{x^2}{\sqrt{10 - x^3}}$

$|g_3'(2)| \approx 2.12 > 1$, and hence the derivative condn fails.

Numerics $\to$ we have convergence; why is it so?

In the interval $[1, 1.5]$: $g_3 : [1, 1.5] \to [1, 1.5]$ and $|g_3'(x)| < 1$.

Why is $g_1(x)$ so good?

Suppose $f \in C^2[a,b]$. Let $p \in (a,b)$ be a root of f. i.e, $f(p)=0$.
and let $p_0 \in [a,b]$ s.t. $f'(p_0) \neq 0$.

Consider the Taylor series approximation of $f(x)$ about p_0 and evaluated at p.

$$\underbrace{f(p)}_{\substack{\parallel \\ 0}} = f(p_0) + \frac{(p-p_0)}{1!} f'(p_0) + \underbrace{\frac{(p-p_0)^2}{2!} f''(\xi)}_{\text{Neglect}}, \quad \xi \in (p_0, p)$$

We assume p is close to p_0, the higher-order terms are neglected

$$-f(p_0) = (p-p_0) f'(p_0)$$

$$\Rightarrow \quad p = p_0 - \frac{f(p_0)}{f'(p_0)}$$

P_1

$\vdots$

$$\boxed{p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}}, \quad n \geq 1$$

Newton-Raphson Method

Consider this as a fixed-point method

$$g(x) = x - \frac{f(x)}{f'(x)}$$

Example : $f(x) = x^2 - 6$; $p_0 = 1$ $\therefore f'(x) = 2x$

Iteration 1: $p_1 = p_0 - \frac{f(p_0)}{f'(p_0)} = 1 - \frac{(-5)}{2} = 3.5$

Rerror : $\frac{|p_1 - p_0|}{|p_1|} = 0.7142857$

Iteration 2 : $p_2 = p_1 - \frac{f(p_1)}{f'(p_1)} \approx 2.607124$

Rel. error ≈ 0.342

$f(x)$ has a zero at p, then

$$f(x) = (x-p)\,h_1(x) \quad \text{for some } h_1(x)$$

$f(x)$ has a double zero at p then

$$f(x) = (x-p)^2 h_2(x)$$

$$f'(x) = 2(x-p)h_2(x) + (x-p)^2 h_2'(x) = (x-p)\left[2h_2(x) + (x-p)h_2'(x)\right]$$

$$f'(p) = 0$$

Modified Newton : Newton of $f \longrightarrow$ Newton on $\dfrac{f(x)}{f'(x)} = \beta(x)$

$$x_{n+1} = x_n - \frac{\beta(x_n)}{\beta'(x_n)}$$

Theorem (Convergence of Newton's Method)

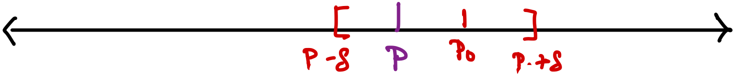
Let $f \in C^2[a, b]$. Suppose that $p \in (a, b)$ satisfies

$$f(p) = 0 \quad \text{and} \quad f'(p) \neq 0.$$

Then there exists a $\delta > 0$ such that Newton's method generates a sequence $\{p_n\}_{n=0}^{\infty}$ converging to p for any initial approximation

$$p_0 \in [p - \delta, p + \delta].$$

$\{p_n\}$



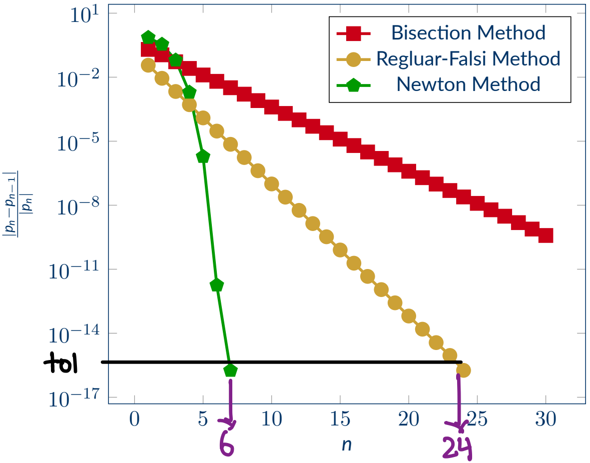


Figure 1: Relative error of the bisection method, regular falsi method, and the Newton method applied to $f(x) = x^2 - 6$ on the interval $[2, 4]$.

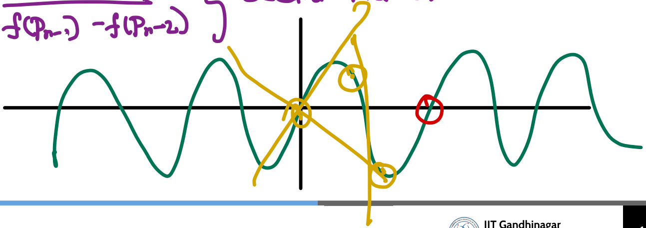
$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$

$$f'(p_{n-1}) = \lim_{x \to p_{n-1}} \frac{f(x) - f(p_{n-1})}{x - p_{n-1}}$$: Approximate this.

$x \to p_{n-2}$

$$\approx \frac{f(p_{n-1}) - f(p_{n-2})}{p_{n-1} - p_{n-2}}$$

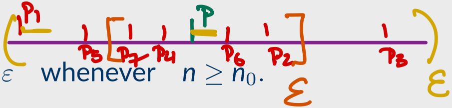
$$p_n = p_{n-1} - f(p_{n-1}) \frac{(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}$$ } Secant Method.



Definition (Convergence of a Sequence)

A sequence $\{p_n\}$ is said to converge to a number p if for every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that

$$|p_n - p| < \varepsilon \quad \text{whenever} \quad n \geq n_0.$$



Definition (Order of Convergence)

Suppose $\{p_n\}_{n=0}^{\infty}$ is a sequence that converges to p, with $p_n \neq p$ for all n. If there exist positive constants λ and α such that

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda, \tag{1}$$

then the sequence $\{p_n\}$ is said to converge to p with order α and asymptotic error constant λ.

Example : $p_n = \frac{1}{2^n}$ $\quad p_n \to ? \, 0$ as $n \to \infty$

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \left| \frac{1/2^{n+1} - 0}{\left(1/2^n - 0\right)^\alpha} \right| = \frac{2^{n\alpha}}{2^{n+1}} = 2^{n\alpha - n - 1}$$

For $\alpha = 1$: $2^{-1} \to \frac{1}{2}$

$\alpha = 2$; $2^{2n-n-1} = 2^{n-1} \to \infty$.

$\Rightarrow$ Bisection method is linearly converging.

Theorem

Let $g \in \mathcal{C}[a,b]$ be such that

$$g(x) \in [a,b] \quad \forall x \in [a,b].$$

Suppose, in addition, that g' is continuous on (a,b) and there exists a constant $0 < k < 1$ such that

$$|g'(x)| \leq k \quad \forall x \in (a,b).$$

If $g'(p) \neq 0$, then for any $p_0 \neq p$ in $[a,b]$, the sequence

To get higher-order convergence, we need to allow $g'(p)=0$

$$p_n = g(p_{n-1}), \quad n \geq 1,$$

converges linearly to the unique fixed point p in $[a,b]$.

Theorem

Let p be a solution of the equation $x = g(x)$. Suppose that $g'(p) = 0$ and that g'' is continuous on an open interval I containing p, with

$$|g''(x)| \leq M \quad \forall x \in I.$$

Quadratic convergence.

Then there exists a $\delta > 0$ such that, for any

$$p_0 \in [p - \delta, p + \delta],$$

the sequence defined by

$$p_n = g(p_{n-1}), \quad n \geq 1,$$

converges at least quadratically to p. Moreover, for sufficiently large n,

$$|p_{n+1} - p| \leq \frac{M}{2} |p_n - p|^2.$$

The above two theorem states

$$g(p) = p \qquad g'(p) = 0$$

As seen earlier, $g(x) = x - \phi(x) f(x)$ where ϕ is differentiable.

We require for quadratic convergence: $g'(p) = 0$

$$g'(x) = 1 - \phi'(x) f(x) - \phi(x) f'(x)$$

at $x = p$:

$$g'(p) = 1 - \phi'(p) \underbrace{f(p)}_{=0} - \phi(p) f'(p)$$

$$0 = 1 - \phi(p) f'(p) \Rightarrow \phi(p) = \frac{1}{f'(p)}$$

This ensures $g'(p) = 0$ and hence quadratic convergence. Therefore we take

$$g(x) = x - \frac{f(x)}{f'(x)}$$

as the fixed-point function.

$$\frac{|P_{n+1}-P|}{|P_n-P|^\alpha}=\lambda \implies |P_{n+1}-P|=\lambda\,|P_n-P|^\alpha$$

Take log on both sides

$$\implies \log(|P_{n+1}-P|)=\log(\lambda)+\alpha\log|P_n-P|$$

$$\implies \alpha=\frac{\log(|P_{n+1}-P|)-\log(\lambda)}{\log(|P_n-P|)}$$

If $\log(\lambda)$ is small $\implies \alpha\approx\dfrac{\log(|P_{n+1}-P|)}{\log(|P_n-P|)}$

x^2-6; $[2,4]$: $p=\sqrt{6}=2.449489744$

Bisection Method			Newton Method		
n	Error	Order α	n	Error	Order α
0	5.103×10^{-1}	–	1	1.051×10^{0}	–
1	4.995×10^{-1}	0.092	2	1.577×10^{-1}	-37.49
2	2.495×10^{-1}	2.000	3	4.767×10^{-3}	2.894
3	1.245×10^{-1}	1.501	4	4.629×10^{-6}	2.298
4	6.199×10^{-2}	1.335	5	4.374×10^{-12}	2.129
5	3.074×10^{-2}	1.252	6	0.000×10^{0}	–
6	1.511×10^{-2}	1.204			
7	7.302×10^{-3}	1.174			
8	3.396×10^{-3}	1.156			
9	1.443×10^{-3}	1.151			
10	4.663×10^{-4}	1.173			

Table 2: Observed order of convergence for the bisection and Newton methods.

Bracketing: Bisection → 1st order ; $p_0 \in [a, b]$
Regular-falsi
Need starting interval/Bracket

Open-Domain: Fixed-point ; Not require interval but p_0
Newton
Secant → 2nd order

Error Analysis: Order of Convergence.