Lecture 4: System of Equations

Abhinav Jha

Indian Institute of Technology, Gandhinagar

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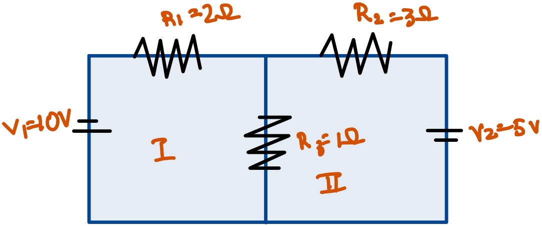
Outline

Direct Methods



Kirchoff's Voltage law:

$V_i = I_i R_i$ (Ohm's law) for $i = 1, 2, 3$

In loop 1

$$10 - 2I_1 - (I_1 - I_2) = 0$$

$$\Rightarrow \boxed{-3I_1 + I_2 = -10}$$

In loop 2 :

$$5 - 3I_2 - (I_2 - I_1) = 0$$

$$\Rightarrow \boxed{I_1 - 4I_2 = -5}$$

I_1, I_2

$$\left.\begin{array}{ll} R_1 & a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ R_2 & a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots & \\ R_n & a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{array}\right\}$$

$\{x_i\}_{i=1}^{n} \to$ Unknowns

Row numbers.

1. Scalar multiplication : $R_i \mapsto \lambda R_i$, for λ real number $\neq 0$
2. Row replacement : $R_i \mapsto R_i + \lambda R_j$ for $j \neq i$
3. Row interchange : $R_i \leftrightarrow R_j$, $i \neq j$

1. Augmented Matrix

$A\underline{x} = \underline{b}$; $A = \{a_{ij}\}$; $\underline{b} = \{b_i\}_{i=1}^{n}$, $\underline{x} = \{x_i\}_{i=1}^{n}$

$b_1 \to a_{1,n+1}$, $b_n \to a_{n,n+1}$

$$[A, \underline{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{array}\right] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & a_{1,n+1} \\ a_{21} & a_{22} & \dots & a_{2n} & a_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & a_{n,n+1} \end{array}\right]$$

$$\begin{array}{c} R_1 \\ R_2 \\ \vdots \\ R_n \end{array} \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & a_{1,n+1} \\ a_{21} & a_{22} & \cdots & a_{2n} & a_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & a_{n,n+1} \end{array}\right] \xrightarrow{\text{How?}} \left[\begin{array}{cccc|c} \bar{a}_{11} & \bar{a}_{12} & \cdots & \bar{a}_{1n} & \bar{a}_{1,n+1} \\ 0 & \bar{a}_{22} & \cdots & \bar{a}_{2n} & \bar{a}_{2,n+1} \\ 0 & 0 & \bar{a}_{33} \cdots & \bar{a}_{3n} & \bar{a}_{3,n+1} \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 \cdots & \bar{a}_{nn} & \bar{a}_{n,n+1} \end{array}\right]$$

$R_2 \to R_2 - \frac{a_{21}}{a_{11}} R_1$ (so that $a_{21} = 0$)

$$\left[\begin{array}{ccccc} a_{11} & a_{12} & \cdots & a_{1n} & a_{1,n+1} \\ 0 & \tilde{a}_{22} & \cdots & \tilde{a}_{2n} & a_{2,n+1} \\ a_{31} & a_{32} & \cdots & a_{3n} & a_{3,n+1} \\ \vdots & \vdots & \ddots & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} & a_{n,n+1} \end{array}\right]$$

$R_j \to R_j - \frac{a_{j1}}{a_{11}} R_1, \quad j = 2, \cdots, n$

my 1st col is $\begin{bmatrix} a_{11} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Moving in this fashion, we get

$$\left[\begin{array}{ccccc|c} \bar{a}_{11} & \bar{a}_{12} & \cdots & \bar{a}_{1n} & & \bar{a}_{1,n+1} \\ 0 & \bar{a}_{22} & \cdots & \bar{a}_{2n} & & \bar{a}_{2,n+1} \\ 0 & 0 & \bar{a}_{33} & \cdots & \bar{a}_{3n} & \bar{a}_{3,n+1} \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \bar{a}_{nn} & \bar{a}_{n,n+1} \end{array}\right] \sim$$

$$\bar{a}_{11}x_1 + \bar{a}_{12}x_2 + \dots + \bar{a}_{1n}x_n = \bar{a}_{1,n+1}$$
$$\bar{a}_{22}x_2 + \dots + \bar{a}_{2n}x_n = \bar{a}_{2,n+1}$$
$$\vdots$$
$$\bar{a}_{n-1,n-1}x_{n-1} + \bar{a}_{n-1,n}x_n = \dots$$
$$\bar{a}_{nn}x_n = \bar{a}_{n,n+1}$$

$$x_n = \frac{\bar{a}_{n,n+1}}{\bar{a}_{nn}}$$

$$\bar{a}_{n-1,n-1}x_{n-1} + \bar{a}_{n-1,n}x_n = \bar{a}_{n-1,n+1}$$

$$x_{n-1} = \frac{\bar{a}_{n-1,n+1} - \bar{a}_{n-1,n}x_n}{\bar{a}_{n-1,n-1}}$$

$\rightarrow$ Backward Substitution

We continue this to get $\{x_i\}_{i=1}^{n}$

Example: $[A,b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 3 & 1 & 11 \\ 3 & 2 & 2 & 13 \end{array}\right]$ $\xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1}$ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & -1 & -5 \end{array}\right]$ $R_3 \to R_3 + R_2$ $= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -2 & -6 \end{array}\right]$

$0 \leftarrow$

With Backward subs: $\underline{x} = [1 \;\; 2 \;\; 3]^T$.

Suppose is 0.

Example: $A = \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -8 \\ 2 & -2 & 3 & -3 & -20 \\ 1 & 1 & 1 & 0 & -2 \\ 1 & -1 & 4 & 3 & 4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -8 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & 2 & 4 & 12 \end{array}\right]$

Pivoting to avoid division by zero → Locate the 1st non-zero entry and then Row interchange; $R_2 \leftrightarrow R_3$

0

$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & 0 & a_{23} & b_2 \\ 0 & 0 & a_{33} & b_3 \end{array}\right]$ → System no solution or ∞ solution

$a_{33} = 0$, $b_3 \neq 0$

Matrix-Reduction

A/S : $\frac{n^3-n}{3}$

M/D : $\frac{2n^3+3n^2-5n}{6}$

$\frac{n^3}{3}+n^2-\frac{n}{3}$

Backward Substitution

A/S : $\frac{n^2-n}{6}$

M/D : $\frac{n^2+n}{2}$

$\frac{n^3}{3}+\frac{n^2}{2}-\frac{5n}{6}$

$$\left[\begin{array}{cccc|c} \bar{a}_{11} & \bar{a}_{12} & \cdots & \bar{a}_{1n} & \bar{a}_{1,n+1} \\ 0 & \bar{a}_{22} & \cdots & \bar{a}_{2n} & \bar{a}_{2,n+1} \\ 0 & 0 & \bar{a}_{33} \cdots & \bar{a}_{3n} & \bar{a}_{3,n+1} \\ \vdots & \vdots & \vdots \ddots & \vdots & \vdots \\ 0 & 0 & 0 \cdots & \bar{a}_{nn} & \bar{a}_{n,n+1} \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} \tilde{a}_{11} & 0 & \cdots & 0 & \tilde{a}_{1,n+1} \\ 0 & \tilde{a}_{22} & \cdots & 0 & \tilde{a}_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \tilde{a}_{nn} & \tilde{a}_{n,n+1} \end{array}\right]$$

$$x_i = \frac{\tilde{a}_{i,n+1}}{\tilde{a}_{ii}}, \text{ for } i = 1, \ldots, n$$

L: Lower triangular matrix

U: Upper triangular matrix

$L\underline{x}=\underline{b}$ or $U\underline{x}=\underline{b}$ $\longrightarrow$ Complexity is $O(n^2)$

$$A = LU \longrightarrow A\underline{x}=\underline{b}$$

$$\Rightarrow L(U\underline{x})=\underline{b}$$

$$\Rightarrow L\underline{y}=\underline{b} \quad \text{where} \quad \underbrace{U\underline{x}=\underline{y}}$$

Step 1: Solve this compute $\underline{y}$ Step 2 : Solve for $\underline{x}$

Assumption: $a_{ii}^{(k)} \neq 0 \ \forall i$, and steps k. (No pivoting)

$A \xrightarrow[\text{elimination}]{\text{Gaussian}} U \to$ Compute L.

$$\begin{bmatrix} a_{11} & & \\ a_{21} & & \\ \vdots & \times & \\ a_{n1} & & \end{bmatrix} \rightsquigarrow \begin{bmatrix} a_{11} & & \\ 0 & & \\ \vdots & \times & \\ 0 & & \end{bmatrix}$$

$R_j \to R_j - m_{j1} R_1$, where $m_{j1} = \dfrac{a_{j1}^{(1)}}{a_{11}^{(1)}}$, $j = 2, \cdots, n$

equivalent to multiplying A with $M^{(1)}$

Same

$$M^{(1)} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -m_{21} & 1 & 0 & \cdots & 0 \\ -m_{31} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -m_{n1} & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$M^{(1)} A = \begin{bmatrix} 1 & 0 & & \\ -m_{21} & 1 & & \\ -m_{31} & 0 & \times & \\ \vdots & \vdots & & \\ -m_{n1} & 0 & & \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \\ a_{21} & a_{22} & \\ \vdots & \vdots & x \\ a_{n1} & a_{n2} & \end{bmatrix}$$

$$= \begin{bmatrix} \\ 0 \\ \\ \end{bmatrix}$$

$-m_{21} a_{11} + a_{21}$

$= -\dfrac{a_{21}}{\cancel{a_{11}}} \cancel{a_{11}} + a_{21} = 0$

$$\underline{A}\underline{x} = \underline{b} \to \underbrace{M^{(1)} A}\,\underline{x} = M^{(1)}\underline{b}$$ → First Gaussian elimination

$$\begin{bmatrix} a_{11} & a_{12} & \cdots \\ 0 & a_{22} & \cdots \\ 0 & a_{32} & \\ \vdots & \vdots & \\ 0 & a_{n2} & \cdots \end{bmatrix}$$

$$M^{(2)} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & -m_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -m_{n2} & 0 & \cdots & 1 \end{bmatrix} \rightsquigarrow \underbrace{M^{(2)} M^{(1)} A}_{A^{(3)}}\,\underline{x} = M^{(2)} M^{(1)} \underline{b}$$

Continuing

$$A^{(n)} = A^{(n)}\underline{b}$$

where $\underbrace{A^{(n)}} = M^{(n-1)} M^{(n-2)} \cdots M^{(1)} A = U$ → Inverse L

What kind of matrix? Upper U

$L^{-1}A = U$

$A = LU$

LU Decomposition

Definition (Diagonally Dominant Matrix)

A matrix $\mathbf{A} = (a_{ij})$ is said to be *diagonally dominant* if

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|, \qquad i = 1, 2, \ldots, n.$$

If the inequality is strict for every i, then $\mathbf{A}$ is called *strictly diagonally dominant*.

Theorem

A strictly diagonally dominant matrix $\mathbf{A}$ is nonsingular. Moreover, Gaussian elimination applied to a system $\mathbf{Ax} = \mathbf{b}$ can be carried out without row or column interchanges, and the computation is stable with respect to the growth of round-off errors.

Definition (**Positive Definite Matrix**)

A matrix $\mathbf{A}$ is said to be *positive definite* if

$$\mathbf{x}^\top \mathbf{A}\mathbf{x} > 0 \quad \text{for all } \mathbf{x} \neq \mathbf{0}.$$

If $\mathbf{A}$ is symmetric and positive definite, it is called a *symmetric positive definite (SPD)* matrix.