Lecture 6: System of Equations

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Definition

A *vector norm* on $\mathbb{R}^n$ is a function $||\cdot|| : \mathbb{R}^n \to \mathbb{R}$ satisfying the following properties:

1. $||\mathbf{x}|| \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$.
2. $||\mathbf{x}|| = 0$ if and only if $\mathbf{x} = \mathbf{0}$.
3. $||\alpha\mathbf{x}|| = |\alpha|\,||\mathbf{x}||$ for all $\alpha \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$.
4. $||\mathbf{x}+\mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

eg. $\underline{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$

then $\|\underline{x}\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$; $\|\underline{x}\|_\infty = \max_{1 \le i \le n} |x_i|$

l_2 or Euclidean norm ; l_∞ or max norm.

Fixed-point : Generated a sequence $\{p_n\}_{n=0}^{\infty}$; $p_n \to p$;

$\forall \varepsilon > 0$, $\exists\, n_0(\varepsilon) \in \mathbb{N}$ s.t. $|p_n - p| < \varepsilon$ whenever $n \geq n_0$.

($\mathbb{N}$: Natural number)

Iterative method : $\{\underline{x}_n\}_{n=0}^{\infty}$; $\underline{x}_n \to \underline{x}$; $\underline{x} \in \mathbb{R}^n$

$\forall \varepsilon > 0$, $\exists\, n_0(\varepsilon) \in \mathbb{N}$ s.t.

$\|\underline{x}_n - \underline{x}\| < \varepsilon$ whenever $n \geq n_0$.

Theorem

If $\|\cdot\|$ is a vector norm on $\mathbb{R}^n$, then

$$\|\mathbf{A}\| = \max_{\|\mathbf{x}\|=1} \|\mathbf{A}\mathbf{x}\|$$

defines a matrix norm.

Example: $\|A\|_\infty = \max_{\|\underline{x}\|_\infty = 1} \|Ax\|_\infty$

1. $\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$. → Induced norm

2. $\|A\|_\infty = \max_{1\le i,j\le n} |a_{ij}|$ → ✗

3. $\|A\|_{Fr} = \sqrt{\text{trace}(A^T A)}$

$= \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{1/2}$

Matrix Norms

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

$$x_1 = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2 - \cdots - a_{1n}x_n\right)$$

$$x_2 = \frac{1}{a_{22}}\left(b_2 - a_{21}x_1 - \cdots - a_{2n}x_n\right)$$

$$\vdots$$

$$x_n = \frac{1}{a_{nn}}\left(b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{n,n-1}x_{n-1}\right)$$

$$x_i = \frac{1}{a_{ii}}\left(b_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}x_j\right)$$

$$x_1^{(k)\,/\,(1)} = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^{(k-1)\,/\,(0)} - \cdots - a_{1n}x_n^{(k-1)\,/\,(0)}\right)$$

$$x_2^{(k)\,/\,(1)} = \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{(k-1)\,/\,(0)} - \cdots - a_{2n}x_n^{(k-1)\,/\,(0)}\right)$$

$$\vdots$$

$$x_n^{(k)\,/\,(1)} = \frac{1}{a_{nn}}\left(b_n - a_{n1}x_1^{(k-1)\,/\,(0)} - a_{n2}x_2^{(k-1)\,/\,(0)} - \cdots - a_{n,n-1}x_{n-1}^{(k-1)\,/\,(0)}\right)$$

$x^{(0)} = (\quad) \in \mathbb{R}^n$; Random vector

$x^{(2)} =$

$x^{(1)}$

Example :

$$4x_1 + x_2 - x_3 = 5$$
$$-x_1 + 3x_2 - x_3 = -4$$
$$2x_1 + 2x_2 + 5x_3 = 1$$

Each $a_{ii} \neq 0 \ \forall i$, then:

$$x_1^{(k)} = \frac{1}{4}\left(5 - x_2^{(k-1)} + x_3^{(k-1)}\right)$$

$$x_2^{(k)} = \frac{1}{3}\left(-4 + x_1^{(k-1)} - x_3^{(k-1)}\right)$$

$$x_3^{(k)} = \frac{1}{5}\left(1 - 2x_1^{(k-1)} - 2x_2^{(k-1)}\right) \quad \text{for } k \geq 1$$

Let $\underline{x}^{(0)} = (0,0,0)$

$\therefore \underline{x}^{(1)} = \left(\frac{5}{4}, \frac{-4}{3}, \frac{1}{5}\right)$; $\underline{x}^{(2)} = (1.133, -0.9833, 0.233)$

In general, iterative method has

$$\underline{x} = T\underline{x} + \underline{c}$$

for some matrix T and vector $\underline{c}$. Then

$$\underline{x}^{(k)} = T\underline{x}^{(k-1)} + \underline{c}$$

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}$$

$$A = D - L - U \; ; \; D = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} ; \; L = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ -a_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & 0 \end{bmatrix} ; \; U = \begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ 0 & 0 & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$A\underline{x} = \underline{b} \Rightarrow (D - L - U)\underline{x} = \underline{b}$$

$$\Rightarrow D\underline{x} = (L+U)\underline{x} + \underline{b}$$

$$\Rightarrow \underline{x} = \underbrace{D^{-1}(L+U)}_{T_J}\underline{x} + \underbrace{D^{-1}\underline{b}}_{\underline{c}_J}$$

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{11}x_1 = b_1 - a_{12}x_2 - \dots - a_{1n}x_n$$

$$\Rightarrow \underline{x}^{(k)} = T_J\underline{x}^{(k-1)} + \underline{c}_J.$$

$$a_{11} x_1^{(k)} + a_{12} x_2^{(k-1)} + \dots + a_{1n} x_n^{(k-1)} = b_1$$

$$a_{21} x_1^{(k-1)\,(k)} + a_{22} x_2^{(k)} + \dots + a_{2n} x_n^{(k-1)} = b_2$$

$$\vdots$$

$$a_{n1} x_1^{(k-1)/(k)} + a_{n2} x_2^{(k-1)(k)} + \dots + a_{nn} x_n^{(k)} = b_n$$

Jacobi

Gauss-Seidel

$$\Rightarrow \quad x_1^{(1)} = \frac{1}{a_{11}} \left(b_1 - a_{12} x_2^{(0)} - \dots - a_{nn} x_n^{(0)} \right)$$

$$x_2^{(1)} = \frac{1}{a_{22}} \left(b_2 - a_{21} x_1^{(1)} - \dots - a_{nn} x_n^{(0)} \right)$$

$$\vdots$$

$$x_n^{(1)} = \frac{1}{a_{nn}} \left(b_n - a_{n1} x_1^{(1)} - \dots - a_{n-1,n} x_{n-1}^{(1)} \right)$$

Example:

$$4x_1 + x_2 - x_3 = 5$$
$$-x_1 + 3x_2 + x_3 = -4$$
$$2x_1 + 2x_2 + 5x_3 = 1$$

$$x_1^{(k)} = \frac{1}{4}\left(5 - x_2^{(k-1)} + x_3^{(k-1)}\right)$$

$$x_2^{(k)} = \frac{1}{3}\left(-4 + x_1^{(k)} - x_3^{(k-1)}\right)$$

$$x_3^{(k)} = \frac{1}{5}\left(1 - 2x_1^{(k)} - 2x_2^{(k)}\right)$$

$$\underline{x}^{(0)} = (0,0,0)$$

$$x_1^{(1)} = \frac{1}{4}(5 - 0 + 0) = 1.25$$

$$x_2^{(1)} = \frac{1}{3}(-4 + 1.25 - 0)$$
$$= -0.9166$$

$$x_3^{(1)} = \frac{1}{5}(1 - 2 \times 1.25 - 2 \times (-0.9166))$$
$$= -0.0666.$$

$$\underline{x}^{(2)} = (1.4958, -0.8569, -0.0551).$$

$$a_{11}x_1^{(k)} + a_{12}x_2^{(k-1)} + \dots + a_{1n}x_n^{(k-1)} = b_1$$

$$a_{21}x_1^{(k)} + a_{22}x_2^{(k)} + \dots + a_{2n}x_n^{(k-1)} = b_2$$

$$\vdots$$

$$a_{n1}x_1^{(k)} + a_{n2}x_2^{(k)} + \dots + a_{nn}x_n^{(k)} = b_n$$

$A = D - L - U$.

$$A\underline{x} = \underline{b} \rightsquigarrow \underline{x}^{(k)} = T\underline{x}^{(k-1)} + \underline{c}$$

$$\Rightarrow (D-L-U)\underline{x} = \underline{b}$$

$$\Rightarrow (D-L)\underline{x} = \underline{b} + U\underline{x}$$

$$\Rightarrow \underline{x} = \underbrace{(D-L)^{-1}U}_{T_{GS}}\underline{x} + \underbrace{(D-L)^{-1}\underline{b}}_{\underline{c}_{GS}}.$$

$$\underline{x}^{(k)} = T_{GS}\underline{x}^{(k-1)} + \underline{c}_{GS}.$$

Theorem (Convergence of Linear Fixed-Point Iteration)

For any initial vector $\mathbf{x}^{(0)} \in \mathbb{R}^n$, *the sequence* $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ *defined by*

$$\mathbf{x}^{(k)} = \mathbf{T}\mathbf{x}^{(k-1)} + \mathbf{c}, \qquad k \geq 1,$$

converges to the unique fixed point $\mathbf{x} = \mathbf{T}\mathbf{x} + \mathbf{c}$ *if and only if*

$$\rho(\mathbf{T}) < 1,$$

where $\rho(\mathbf{T})$ *denotes the spectral radius of* $\mathbf{T}$.

Defn: Spectral Radius

$$\rho(A) = \max_{\lambda \in \sigma(A)} |\lambda|,$$ where $\sigma(A)$ = Set of all eigenvalues of A.

Corollary (Error Bounds)

If $\|\mathbf{T}\| < 1$ for some induced (natural) matrix norm and $\mathbf{c}$ is a given vector, then for any $\mathbf{x}^{(0)} \in \mathbb{R}^n$, the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ defined by

$$\mathbf{x}^{(k)} = \mathbf{T}\mathbf{x}^{(k-1)} + \mathbf{c}$$

converges to the unique fixed point $\mathbf{x} = \mathbf{T}\mathbf{x} + \mathbf{c}$. Moreover, the following error bounds hold:

1. $$\|\mathbf{x} - \mathbf{x}^{(k)}\| \leq \|\mathbf{T}\|^k \, \|\mathbf{x}^{(0)} - \mathbf{x}\|.$$

2. $$\|\mathbf{x} - \mathbf{x}^{(k)}\| \leq \frac{\|\mathbf{T}\|^k}{1 - \|\mathbf{T}\|} \, \|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|.$$

Theorem (Diagonal Dominance)

If the matrix $\mathbf{A}$ *is strictly diagonally dominant, then for any initial guess* $\mathbf{x}^{(0)} \in \mathbb{R}^n$*, both the Jacobi and the Gauss–Seidel methods generate sequences* $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ *that converge to the unique solution of the system* $\mathbf{Ax} = \mathbf{b}$.

Example:

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 2 & 7 & 1 \\ 1 & -3 & 12 \end{bmatrix}; \quad \underline{b} = \begin{bmatrix} 6 \\ 19 \\ 31 \end{bmatrix}$$

$$; \underline{x} = [0.16346 \quad 2.2211 \quad 3.125].$$

Convergence Analysis

Gauss–Seidel Method		Jacobi Method	
Iteration	Relative Error	Iteration	Relative Error
0	3.0297619047619047	0	2.7142857142857144
1	1.3288690476190474	1	1.3244047619047619
2	1.1076566043083935e-02	2	2.9931972789115680e-01
3	3.1308181444700267e-03	3	6.9745606575963850e-02
4	2.6591374897289020e-04	4	1.6258326247165610e-02
5	1.0467338035224927e-05	5	4.1876931824313960e-03
6	1.0184724974493520e-07	6	1.7016688557863757e-03
7	3.2089560897397010e-08	7	3.5870013378047627e-04
8	2.3501450785445854e-09	8	1.2695183670297094e-04
9	7.9357742599484030e-11	9	4.5606091728322970e-05
		10	1.2516623709157848e-05
		11	4.4680801379870520e-06
		12	1.3864473483726770e-06
		13	4.3891857504085863e-07
		14	1.4488958421932097e-07
		15	4.5451457375556004e-08
		16	1.4724687469325204e-08
		17	4.7262667113301404e-09
		18	1.5090069194201305e-09
		19	4.8637660565731270e-10
		20	1.5566214983664395e-10
		21	4.9958814862804957e-11

Theorem

Suppose that $a_{ij} \leq 0$ for all $i \neq j$ and $a_{ii} > 0$ for $i = 1, 2, \ldots, n$. Then exactly one of the following cases holds:

1. $$0 \leq \rho(\boldsymbol{T}_{\mathrm{GS}}) < \rho(\boldsymbol{T}_{\mathrm{J}}) < 1,$$
2. $$1 < \rho(\boldsymbol{T}_{\mathrm{J}}) < \rho(\boldsymbol{T}_{\mathrm{GS}}),$$
3. $$\rho(\boldsymbol{T}_{\mathrm{J}}) = \rho(\boldsymbol{T}_{\mathrm{GS}}) = 0,$$
4. $$\rho(\boldsymbol{T}_{\mathrm{J}}) = \rho(\boldsymbol{T}_{\mathrm{GS}}) = 1.$$

$A\underline{x} = \underline{b}$ (Assume A is invertible)

Suppose $\underline{b}$ is $\underline{b} + \delta\underline{b}$ then $\underline{x}$ has error, say $\delta\underline{x}$:

$$A(\underline{x} + \delta\underline{x}) = \underline{b} + \delta\underline{b}$$

$$\Rightarrow A\underline{x} + A\delta\underline{x} = \underline{b} + \delta\underline{b}$$

($A\underline{x} = \underline{b}$)

$$\Rightarrow A\delta\underline{x} = \delta\underline{b} \Rightarrow \delta\underline{x} = A^{-1}\delta\underline{b}$$

Taking norms $\|\delta\underline{x}\| = \|A^{-1}\delta\underline{b}\|$;

$$\leq \|A^{-1}\| \cdot \|\delta\underline{b}\|.$$

$\|A\underline{x}\| \leq \|A\| \cdot \|\underline{x}\|$

$\|\underline{b}\| \leq \|A\| \cdot \|\underline{x}\|$

$$\Rightarrow \frac{\|\delta\underline{x}\|}{\|\underline{x}\|} \leq \frac{\|A^{-1}\|}{\|\underline{x}\|} \|\delta\underline{b}\| \quad \Big] \rightarrow \text{Rel. error.}$$

$$\Rightarrow \frac{1}{\|\underline{x}\|} \leq \frac{\|A\|}{\|\underline{b}\|}$$

$$\leq \underbrace{\|A^{-1}\| \cdot \|A\|}_{K_A} \left(\frac{\|\delta b\|}{\|b\|} \right)$$

$\Rightarrow$ Rel error $(\underline{x}) \leq K_A$ Rel. error $(\underline{b})$

Defn: Condition Number: $K_A = \|A\| \cdot \|A^{-1}\|$ is condn no.

This checks the sensitivity of the soln.

→ If $K_A \approx 1$; then well-conditioned system

→ If $K_A \gg 1$; then ill-conditioned system; Small change in $\underline{b}$ gives large error in $\underline{x}$.

$A = \begin{bmatrix} 1 & 2 \\ 1.0001 & 2 \end{bmatrix}$; $A^{-1} = \begin{bmatrix} -10000 & 10000 \\ 5000.5 & -5000 \end{bmatrix}$

$\|A\|_\infty = 3.0001$; $\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$

$\|A^{-1}\|_\infty = 20000$

Hence, $K_A = 60002$.

Instead of $A\underline{x} = \underline{b}$: $P^{-1}A\underline{x} = P^{-1}\underline{b}$ } Left PC.

where P = Preconditioner.

$AP^{-1}\underline{y} = \underline{b}$; $\underline{x} = P^{-1}\underline{y}$ } Right PC