Lecture 7: Interpolation

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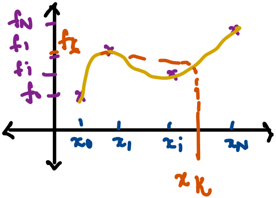
Outline

Figure 1: Elías García Martínez, *Ecce Homo*. The leftmost photograph (2010) shows early flaking of the paintwork. The central photograph (July 2012) shows the extent of deterioration before restoration. The rightmost photograph shows the artwork after the attempted restoration by Cecilia Giménez.

Theorem (Weierstrass Approximation Theorem)

Let $f \in \mathcal{C}[a,b]$. Then for each $\varepsilon > 0$ there exists a polynomial $p(x)$ such that

$$|f(x) - p(x)| < \varepsilon \qquad \text{for all } x \in [a,b].$$



$\{(x_i, f_i)\}_{i=0}^{N}$

$x_k \rightarrow f_k$?

Theorem (Taylor's Theorem))

Suppose $f \in \mathcal{C}^n[a,b]$, $f^{(n+1)}$ exists on $[a,b]$, and $x_0 \in [a,b]$. Then for every $x \in [a,b]$ there exists a number $\xi(x) \in [x_0, x]$ such that

$$f(x) = P_n(x) + R_n(x),$$

Only good around x_0 and not other points

where

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

and

$$R_n(x) \boxed{= \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \underline{(x - x_0)}^{n+1}.}$$

→ Value around x_0.

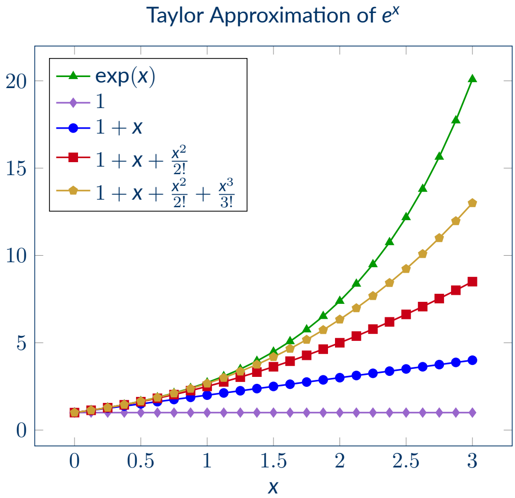
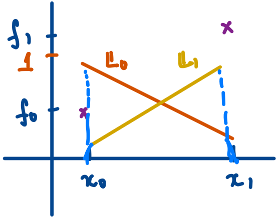


Figure 2: Taylor polynomials for the exponential function expanded about $x = 0$.

$\{(x_0, f_0), (x_1, f_1)\}$

Degree

$\mathbb{L}_0^1(x) = \qquad x_i \qquad \mathbb{L}_1^1(x) =$



$\mathbb{L}_0^1(x_0) = 1 \quad ; \quad \mathbb{L}_1^1(x_0) = 0$

$\mathbb{L}_0^1(x_1) = 0 \quad ; \quad \mathbb{L}_1^1(x_1) = 1$

$$\mathbb{L}_0^1(x) = \frac{x - x_1}{x_0 - x_1} \quad ; \quad \mathbb{L}_1^1(x) = \frac{x - x_0}{x_1 - x_0}$$

$P_1^{\mathbb{L}}(x) \Rightarrow$

$$\left.\begin{aligned} P_1^{\mathbb{L}}(x_0) &= f_0 \\ P_1^{\mathbb{L}}(x_1) &= f_1 \end{aligned}\right\} \rightarrow \boxed{P_1^{\mathbb{L}}(x) = f_0 \underbrace{\mathbb{L}_0^1(x)}_{1,\ 0} + f_1 \underbrace{\mathbb{L}_1^1(x)}_{0,\ 1}}$$

$P_1^{\mathbb{L}}(x_0) = f_0$

$P_1^{\mathbb{L}}(x_1) = f_1$

$$\mathbb{L}_0^1(x) = \frac{x - x_1}{x_0 - x_1}, \quad \mathbb{L}_1^1(x) = \frac{x - x_0}{x_1 - x_0}$$

$$\{(x_i, f_i)\}_{i=0}^{n}$$

$$P_n^{\mathbb{L}}(x) = \sum_{i=0}^{n} f_i \, \mathbb{L}_i^n(x) \rightarrow \text{How?}$$

, c_0

$$\boxed{P_1^{\mathbb{L}}(x) = f_0 \underbrace{\mathbb{L}_0^1(x)}_{} + f_1 \underbrace{\mathbb{L}_1^1(x)}_{0}}$$

Two properties

1. $\mathbb{L}_i^n(x_i) = 1$

2. $\mathbb{L}_i^n(x_j) = 0$, $i \neq j$

Same pattern

$$\boxed{\mathbb{L}_i^n(x) = \frac{(x - x_0)(x - x_1)\cdots(x - x_{i-1})(x - x_{i+1})\cdots(x - x_n)}{(x_i - x_0)(x_i - x_1)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_n)}}$$

$$= \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)}$$

Lagrange Interpolants

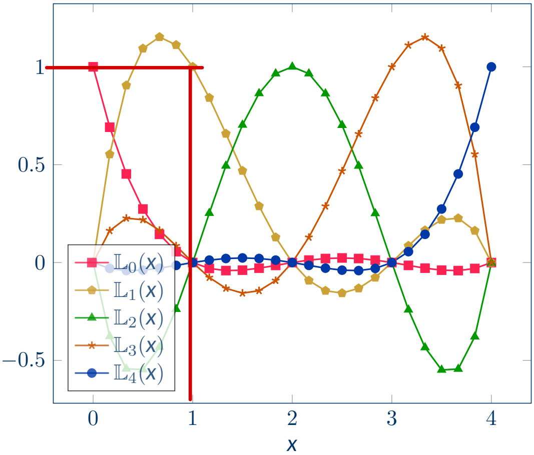


Figure 3: Lagrange basis polynomials defined over $x_i = 0, 1, 2, 3, 4$.

Example : $\{(x_i, f_i)\}_{i=0}^{2}$; $x_0 = 0$, $x_1 = 1$, $x_2 = 2$

$f_0 = 3$, $f_1 = 0$, $f_2 = 8$

Degree $= 2$: $\mathbb{L}_0^2(x)$; $\mathbb{L}_1^2(x)$; $\mathbb{L}_2^2(x)$

$$\mathbb{L}_0^2(x) = \prod_{\substack{j=0\\ j\neq 0}}^{2} \frac{(x-x_j)}{(x_0-x_j)} = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-1)(x-2)}{(0-1)(0-2)} = \frac{1}{2}(x-1)(x-2).$$

$$\mathbb{L}_1^2(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = -x(x-2)$$

$$\mathbb{L}_2^2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{x(x-1)}{2} \rightarrow \frac{(x-0)(x-1)}{(2)(1-0)} = \frac{1}{2}x(x-1)$$

$$P_2^L(x) = f_0\mathbb{L}_0^2(x) + f_1\mathbb{L}_1^2(x) + f_2\mathbb{L}_2^2(x)$$

$$= \frac{3}{2}(x-1)(x-2) + 8x(x-1)$$

Theorem

Suppose $\{x_0, x_1, \ldots, x_n\}$ are distinct points in the interval $[a, b]$ and $f \in \mathcal{C}^{n+1}[a, b]$. Then for each $x \in [a, b]$ there exists a number $\xi(x) \in (a, b)$ such that

$$f(x) = p_n^{\mathbb{L}}(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \boxed{\prod_{i=0}^{n}(x - x_i).} \tag{1}$$

Taylor : Error centered at x_0

Lagrange : Error at all points

Drawbacks

1. Complexity : $\rightarrow$ $O(n^2+n)$

2. Addition of new point $\rightarrow$ $\{(x_{n+1}, f_{n+1})\}$ $\rightarrow$ Add this to data.

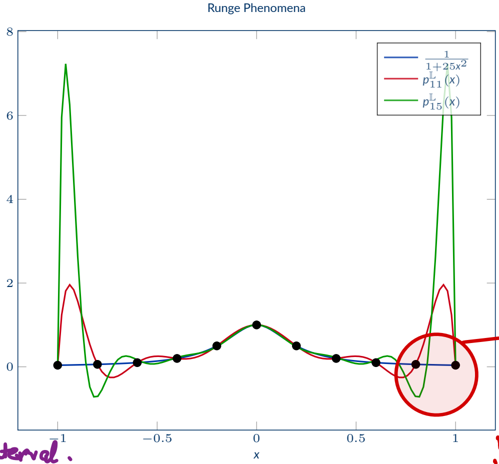
3. Runge Phenomena

$$f(x) = \frac{1}{1+25x^2}, \qquad x \in [-1, 1]$$

Carl Runge :

$$L_i(x_j) = \delta_{ij}$$

$$\delta_{ij} = \begin{cases} 1 & , i = j \\ 0 & , i \neq j \end{cases} \rightarrow \text{Kronecker notation.}$$



Use piecewise interpolation

$[a,b] \to [x_i, x_{i+1}]$

Interpolate on sub-interval.

→ To overcome this put more points on end, i.e. use non-equidistant interpolation.

Figure 4: The polynomial $p^{\mathbb{L}}_{11}(x)$ is computed using 11 equally spaced points (shown as dots), while $p^{\mathbb{L}}_{15}(x)$ uses 15 points.

$\{(x_0, f_0), (x_1, f_1)\}$

$P_1^N(x) = a_0 + a_1(x - x_0)$ where a_0, a_1 are undetermined

$P_1^N(x_0) = f_0 \Rightarrow a_0 = f_1$

$P_1^N(x_1) = f_1 \Rightarrow f_1 = a_0 + a_1(x_1 - x_0)$

$\Rightarrow a_1 = \dfrac{f_1 - f_0}{x_1 - x_0}$

$$P_1^N(x) = \overset{a_0}{f_0} + \underbrace{\overset{a_1}{\frac{f_1 - f_0}{x_1 - x_0}}}_{1^{st}\,DD}(x - x_0)$$

$$P_n^N(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \dots + a_n \underbrace{\prod_{i=0}^{n-1}(x - x_i)}_{\omega_{n-1}(x)}.$$

Divided Difference

x_i	f_i (0^{th} DD)	1^{st} DD	2^{nd} DD	3^{rd} DD ------
x_0	$f_0 = f[x_0]$ a_0	$\frac{f_1 - f_0}{x_1 - x_0} = f[x_0, x_1]$ a_1	a_2 $\frac{f[x_1,x_2] - f[x_0,x_1]}{x_2 - x_0} = f[x_0,x_1,x_2]$	a_3 $\frac{f[x_1,x_2,x_3] - f[x_0,x_1,x_2]}{x_3 - x_0} = f[x_0,x_1,x_2,x_3]$
x_1	f_1	$\frac{f_2 - f_1}{x_2 - x_1} = f[x_1, x_2]$	$\frac{f[x_2,x_3] - f[x_1,x_2]}{x_3 - x_1} = f[x_1,x_2,x_3]$	
x_2	f_2	$\frac{f_3 - f_2}{x_3 - x_2} = f[x_2, x_3]$		
x_3	f_3			

$$P_n^N(x) = f[x_0] + f[x_0, x_1](x - x_0) + \cdots$$

$$= f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k](x - x_0)(x - x_1)\cdots(x - x_{k-1}).$$

→ Example

$$= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

$$+ f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)$$

Example

$x_0=0, \quad x_1=1, \quad x_2=2$

$f_0=3, \quad f_1=0, \quad f_2=8$

x_0	$f_0 = 0^{th}$ DD	1^{st} DD	2^{nd} DD
0	3	$\frac{0-3}{1-0}=-3$	$\frac{8+3}{2-0}=\frac{11}{2}$
1	0	$\boxed{\cancel{\frac{8-0}{2-1}=8}}$	
2	8		

$P_2^N(x) = f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1)$

$= 3 + (-3)(x-0) + \frac{11}{2}(x-0)(x-1)$

$= 3 - 3x + \frac{11}{2}x(x-1)$

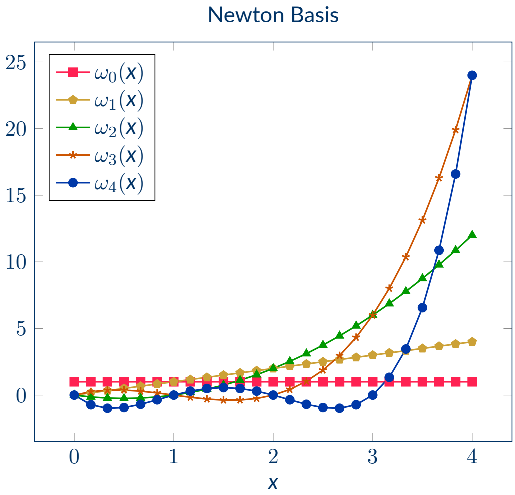


Figure 5: Newton basis polynomials defined over $x_i = 0, 1, 2, 3, 4$.

Newton Divided Difference

Theorem

For distinct points $x_0, \ldots, x_n$, the n^{th} divided difference satisfies

$$f[x_0, x_1, \ldots, x_n] = \sum_{k=0}^{n} \frac{f(x_k)}{\prod_{i \neq k}(x_k - x_i)},$$

with $f[x_0] = f(x_0)$ for $n = 0$.

Advantage

→ Computational complexity → $O(n)$

$$p^N(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots + a_n(x-x_0)\cdots(x-x_{n-1})$$

$$= a_0 + (x-x_0)\left[a_1 + (x-x_1)\left\{a_2 + \dots + (x-x_{n-1})\left(a_{n-1} + a_n(x-x_{n-1})\right)\right\}\right]$$

→ Horner Notation

→ Addition of new points: (x_{n+1}, f_{n+1}).