Lecture 9: Eigenvalue Problems

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Outline

Definition: Let A be a $n \times n$ matrix and let λ be a scalar. If the

$$A\underline{x} = \lambda \underline{x} \qquad (1)$$

has a non-trivial solution, i.e., $\underline{x} \neq 0$ then λ is an eigenvalue and $\underline{x}$ is the eigenvector of A.

eg: $A = \begin{bmatrix} 2 & 0 & 1 \\ 5 & -1 & 2 \\ -3 & 2 & -5/4 \end{bmatrix}$ then $\lambda = -2$ is an e.value

$\underline{x} = [1, 3, -4]$ is an e.vector

Remark: If $(\lambda, \underline{x})$ is an eigenpair then $(\lambda, \alpha \underline{x})$ is also an eigenpair $\forall\, \alpha \neq 0$.

$\rightarrow A(\alpha \underline{x}) = \alpha A \underline{x} = \alpha \lambda \underline{x} = \lambda(\alpha \underline{x})$

Computation: $A\underline{x} = \lambda \underline{x} \Rightarrow A\underline{x} - \lambda I \underline{x} = 0$ where I = Identity matrix

$$\Rightarrow (A - \lambda I)\underline{x} = 0$$

This has non-zero solution if $\det(A - \lambda I) = 0$ (2)

$\det(A - \lambda I) \to$ Polynomial in λ called as the characteristic polynomial.

Roots are the eigenvalues. Direct method.

Power method is used for maximum e-value.

Assumptions: I. There is an unique maximum e-value. ie,

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq, \ldots .$$

II. A has n-linearly independent eigenvectors.

$\{\underline{u}^{(1)}, \underline{u}^{(2)}, \ldots, \underline{u}^{(n)}\} \subseteq \mathbb{R}^n$, s.t. $A\underline{u}^{(i)} = \lambda \underline{u}^{(i)}$ for $i = 1, \ldots, n$.

III. Let $\underline{x}^{(0)} \in \mathbb{R}^n$, then $\underline{x}^{(0)} = \sum_{i=1}^{n} c_i \underline{u}^{(i)}$ for scalar c_i, then $c_1 \neq 0$.

Power Method: Given $\underline{x}^{(0)} \in \mathbb{R}^n$,

$$\underline{x}^{(1)} = A\underline{x}^{(0)}$$

iteratively,

$$\underline{x}^{(2)} = \underbrace{A^2\underline{x}^{(0)}}, \ldots, \quad \underline{x}^{(k)} = A^k\underline{x}^{(0)}.$$

$$\underline{x}^{(2)} = A\underline{x}^{(1)}$$
$$= A(A\underline{x}^{(0)}) = A^2\underline{x}^{(0)}$$

Now; $\underline{x}^{(0)} = \sum_{i=1}^{n} c_i \underline{u}^{(i)}$ for $c_i \in \mathbb{R}$ as $\{\underline{u}^{(i)}\}$ are L.I.

$= \sum_{i=1}^{n} \underline{u}^{(i)}$; as if $\underline{u}^{(i)}$ is e-vector we use is $c_i\underline{u}^{(i)}$

Applying A^k to both sides

$$A^k \underline{x}^{(0)} = \sum_{i=1}^{n} A^k \underline{u}^{(i)}$$

$$\Rightarrow \quad \underline{x}^{(k)} = \sum_{i=1}^{n} A^k \underline{u}^{(i)}$$

$\underline{u}^{(i)}$ is an e-vector of A, i.e. $A\underline{u}^{(i)} = \lambda_i \underline{u}^{(i)} \quad \forall i$

$$\Rightarrow A^2 \underline{u}^{(i)} = \lambda_i A\underline{u}^{(i)}$$
$$= \lambda_i^2 \underline{u}^{(i)}$$
$$\vdots$$
$$A^K \underline{u}^{(i)} = \lambda_i^K \underline{u}^{(i)}$$

Substitute this $\underline{x}^{(K)} = \sum_{i=1}^{n} A^K \underline{u}^{(i)}$

$$= \sum_{i=1}^{n} \lambda_i^K \underline{u}^{(i)} = \lambda_1^K \left[\underline{u}^{(1)} + \sum_{i=2}^{n} \left(\frac{\lambda_i}{\lambda_1}\right)^K \underline{u}^{(i)}\right]$$

Since $|\lambda_1| > |\lambda_i| \; \forall i = 2, \dots, n \Rightarrow \left(\frac{\lambda_i}{\lambda_1}\right)^K \to 0$ as $K \to \infty$

Let $\underline{\varepsilon}^{(K)} = \sum_{i=2}^{n} \left(\frac{\lambda_i}{\lambda_1}\right)^K \underline{u}^{(i)}$. $\longrightarrow \left(\frac{\lambda_2}{\lambda_1}\right)^K$

$$\Rightarrow \underline{x}^{(K)} = \lambda_1^K \left(\underline{u}^{(1)} + \underline{\varepsilon}^{(K)}\right) \quad \text{—} (2)$$

Let $\phi: \mathbb{R}^n \to \mathbb{R}$ be a linear function (functional: $\phi: \mathbb{R}^n \to \mathbb{R}$)

$\phi(\underline{u}^{(1)}) \neq 0$

Then
$$\phi(\underline{x}^{(k)}) = \phi\left(\lambda_1^{(k)}(\underline{u}^{(1)} + \varepsilon^{(k)})\right)$$
$$= \lambda_1^{(k)}\left[\phi(\underline{u}^{(1)}) + \phi(\varepsilon^{(k)})\right] \quad \text{as } \phi \text{ is linear}$$

$[\phi(\alpha x + y) = \alpha\phi(x) + \phi(y)]$

Define
$$r_k = \frac{\phi(\underline{x}^{(k+1)})}{\phi(\underline{x}^{(k)})} = \lambda_1\left[\frac{\cancel{\phi(\underline{u}^{(1)})} + \phi(\varepsilon^{(k+1)})}{\cancel{\phi(\underline{u}^{(1)})} + \phi(\varepsilon^{(k)})}\right]$$

$\phi(\varepsilon^{(k+1)}) \to 0$, $\phi(\varepsilon^{(k)}) \to 0$

Let $k \to \infty$, $r_k \to \lambda_1$. This is the Power method.

As $k \to \infty$, then from Eq. (2) $\underline{x}^{(k)}$ aligns with $\underline{u}^{(1)}$, with a factor of λ_1^k.

The issue happens if $|\lambda_1| > 1$ or $|\lambda_1| < 1$. To solve this problem

$$\underline{x}^{(k)} \leftarrow \frac{\underline{x}^{(k)}}{\|\underline{x}^{(k)}\|}$$

for any norm $\|\cdot\|$, but usually $\|\cdot\|_\infty$

Example : $A = \begin{bmatrix} 6 & 5 & -5 \\ 2 & 6 & -2 \\ 2 & 5 & -1 \end{bmatrix}$, $\underline{x}^{(0)} = (-1, 1, 1)$

$\rightarrow$ Iteration 1 : $\underline{x}^{(1)} = A\underline{x}^{(0)}$

$$= \begin{bmatrix} 6 & 5 & -5 \\ 2 & 6 & -2 \\ 2 & 5 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \\ 2 \end{bmatrix}.$$

$\phi : \mathbb{R}^3 \to \mathbb{R}$. $\phi(\underline{x}) = x_2$. Then

$$v_0 = \frac{\phi(\underline{x}^{(1)})}{\phi(\underline{x}^{(0)})} = \frac{2}{1} = 2$$

Iteration 2: Normalize $\underline{x}^{(1)}$:

$$\|\underline{x}^{(1)}\|_\infty = 6; \quad \underline{x}^{(1)} \leftarrow \left[-1, \tfrac{1}{3}, \tfrac{1}{3}\right]$$

Then $\underline{x}^{(2)} = A\underline{x}^{(1)}$

$$= \begin{bmatrix} -6 \\ 2/3 \\ 2/3 \end{bmatrix}$$

Hence, $\lambda_1 = \dfrac{\phi(\underline{x}^{(2)})}{\phi(\underline{x}^{(1)})} = \dfrac{2/3}{1/3} = 2$

$\downarrow$ We proceed in the same manner until

$$\|\underline{x}^{(k)} - \underline{x}^{(k-1)}\| < tol.$$

Remark : In pratice

$$\phi(\underline{x}) = x_p \text{ , where } x_p = \|\underline{x}\|_\infty .$$

Remark : The convergence of power method relies on

$$\left(\frac{\lambda_i}{\lambda_1}\right)^k \quad \forall i = 2, \ldots, n$$

Since $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \cdots \geq |\lambda_n|$, then

$$|\sigma_k - \lambda_1| \approx \left|\frac{\lambda_2}{\lambda_1}\right|^k .$$

$$\Rightarrow \lim_{k\to\infty} \frac{\overbrace{|\sigma_{k+1} - \lambda_1|}^{e_{k+1}}}{\underbrace{|\sigma_k - \lambda_1|}_{e_k}} \approx \left|\frac{\lambda_2}{\lambda_1}\right| < 1 .$$

converge linearly (See Lec.3)

Example : $\{-1, -2, -3\}$, this converges to $|-3|$, not -1.

$\underline{x}^{(0)} = (-1, -2, -1)$

Iteration	Eigenvalue	x_1	x_2	x_3	Error
0	6.0000	-0.91666667	-1.00000000	-0.91666667	1.0000
1	6.0000	-0.98611111	-1.00000000	-0.98611111	6.9444×10^{-2}
2	6.0000	-0.99768519	-1.00000000	-0.99768519	1.1574×10^{-2}
3	6.0000	-0.99961420	-1.00000000	-0.99961420	1.9290×10^{-3}
4	6.0000	-0.99993570	-1.00000000	-0.99993570	3.2150×10^{-4}
5	6.0000	-0.99998928	-1.00000000	-0.99998928	5.3584×10^{-5}
6	6.0000	-0.99999821	-1.00000000	-0.99999821	8.9306×10^{-6}
7	6.0000	-0.99999970	-1.00000000	-0.99999970	1.4884×10^{-6}
8	6.0000	-0.99999995	-1.00000000	-0.99999995	2.4807×10^{-7}
9	6.0000	-0.99999999	-1.00000000	-0.99999999	4.1345×10^{-8}
10	6.0000	-1.00000000	-1.00000000	-1.00000000	6.8909×10^{-9}

Table 1: Convergence history of the Power Method

Theorem

*If λ is an eigenvalue of **A** and **A** is invertible, then λ^{-1} is an eigenvalue of $\mathbf{A}^{-1}$.*

Proof: Let $(\lambda, \underline{x})$ be an eigenpair, then

$$A\underline{x} = \lambda \underline{x}$$

Since A is invertible $\Rightarrow A^{-1}A\underline{x} = \lambda A^{-1}\underline{x}$

$$\Rightarrow A^{-1}\underline{x} = \frac{1}{\lambda}\underline{x} \Rightarrow (\lambda^{-1}, \underline{x}) \text{ is eigenpair of } A^{-1}.$$

Let eigenvalue of A are paired by

$$|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_{n-1}| > |\lambda_n| > 0$$

$\Rightarrow$ A has unique eigenvalue of smallest order.

$$|\lambda_n^{-1}| > |\lambda_{n-1}^{-1}| \geq \dots \geq |\lambda_2^{-1}| \geq |\lambda_1^{-1}|.$$

Apply power method to A^{-1}; then it computes $|\lambda_n^{-1}|$ (max of A^{-1}) and hence smallest for A.

$$\underline{x}^{(k+1)} = A^{-1}(\underline{x}^{k}) \quad , k \geq 0$$

Computing A^{-1} is inefficient and prone to error. ∴ Hence

$$A\underline{x}^{(k+1)} = \underline{x}^{(k)}.$$

$\rightarrow$ LU decomposition of A and then solve it.

Example: $A = \begin{bmatrix} 6 & 5 & -5 \\ 2 & 6 & -2 \\ 2 & 5 & -1 \end{bmatrix}$; $L = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 1/3 & 10/13 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 6 & 5 & -5 \\ 0 & 13/3 & -1/3 \\ 0 & 0 & 12/13 \end{bmatrix}$.

The IPM, is implemented to $\underline{x}^{(0)} = [-1, 1, 1]$

$$A\underline{x}^{(k+1)} = \underline{x}^{(k)}.$$

→ Iteration 1:

$$A\underline{x}^{(1)} = \underline{x}^{(0)}$$

$\Rightarrow$ $L\underline{y} = \underline{x}^{(0)}$ and $U\underline{x}^{(1)} = \underline{y}$

$\downarrow$ $\qquad\qquad$ $\downarrow$

$\underline{y} = [-1, 4/3, 4/13]$ $\qquad$ $\underline{x}^{(1)} = [-1/6, 1/3, 1/3]$

Let $\phi(x) = \|x\|_\infty$; $\mu_1 = \dfrac{\phi(\underline{x}^{(1)})}{\phi(\underline{x}^{(0)})} = \dfrac{1/3}{1} = 1/3.$

Iteration 2 : Normalize $\underline{x}^{(1)}$; $\underline{x}^{(1)} \leftarrow [-1/2, 1, 1]$

$$A\underline{x}^{(2)} = \underline{x}^{(1)}$$

$$\Rightarrow \underline{x}^{(2)} = [-1/12, 7/24, 7/24]. \quad \mu_1 = \frac{7/24}{1} = 7/24.$$

Remark : $\{-1, 2, -3\}$ .. IPM $\{1\}$.

Suppose E-value $(A) = \{6, 4, 1\}$

we want to compute e-value closest to 3, i.e., 4

→ let λ_j be the e-value closest to μ. i.e $\exists$ eigenvalue

$|\lambda_j - \mu| < \varepsilon$ for some ε.

Consider, the matrix $A - \mu \underline{I}$. If A has e-values $\{\lambda_j\}_{j=1}^{n}$ then

$A - \mu I$ has $\{\lambda_j - \mu\}_{j=1}^{n}$.

__Ex:__ If $\{\lambda_i\}$ are e-value of A; then $\{\lambda_j - \mu\}$ are eigenvals of $A - \mu \underline{I}$

Since, $\lambda_j - \mu \neq 0$. $A - \mu I$ is invertible.

SPM is IPM applied to $A - \mu I$. as it gives the smallest e-value say z^{-1}.

$$\Rightarrow z^{-1} = \lambda_j - \mu$$

$\Rightarrow \lambda_j = z^{-1} + \mu$ is the evalu closest to μ.

$\rightarrow$ SVD $\rightarrow$ Singular Value Decomposition.