Dr. A. Jha

MA643 Numerical Analysis of PDEs

Summer Semester 2025

20.08.2025

Assigment 1

Assignment must be completed in groups of 2 members and submitted via Google Classroom by 03.09.2025 at 23:59. If the assignment is not prepared using LaTeX, a clear scanned copy of the handwritten work must be uploaded. Ensure that the names and enrollment numbers of all group members are clearly written on the submission. Late submissions will not be accepted.

Question 1: [Vector Operations, 4 Points]

The following vector calculus operations are defined for a scalar function $u : \mathbb{R}^3 \to \mathbb{R}$ and a vector field $\mathbf{v} = (v_1, v_2, v_3)^\top : \mathbb{R}^3 \to \mathbb{R}^3$, using the nabla operator

$$\nabla = \begin{bmatrix} \partial_x & \partial_y & \partial_z \end{bmatrix}^\top .$$

- Gradient:
$$\nabla u = \begin{bmatrix} \partial_x u & \partial_y u & \partial_z u \end{bmatrix}^\top .$$
- Divergence:
$$\nabla \cdot \mathbf{v} = \partial_x v_1 + \partial_y v_2 + \partial_z v_3.$$
- Curl:
$$\nabla \times \mathbf{v} = \begin{bmatrix} \partial_y v_3 - \partial_z v_2 & \partial_z v_1 - \partial_x v_3 & \partial_x v_2 - \partial_y v_1 \end{bmatrix}^\top .$$

Assuming all functions are sufficiently smooth, prove the following identities:

a) $\nabla \cdot \nabla u = \Delta u = \partial_{xx} u + \partial_{yy} u + \partial_{zz} u.$

b) $\nabla \cdot (\nabla \times \mathbf{v}) = 0.$

c) $\nabla \times (\nabla u) = \mathbf{0}.$

d) $\nabla \times (u\mathbf{v}) = u(\nabla \times \mathbf{v}) + (\nabla u \times \mathbf{v}).$

e) $\nabla \cdot (u\mathbf{v}) = (\nabla u) \cdot \mathbf{v} + u(\nabla \cdot \mathbf{v}).$

Question 2: [Classification of PDEs, 4 Points]

Classify the following partial differential equations (PDEs) into elliptic, parabolic, and hyperbolic equations.

a) $u_{xx} + 2u_{xy} + 2u_{yy} + 4u_{yz} + 5u_{zz} + u_x + u_y = 0.$

b) $u_{xx} - e^z\, u_{xy} + \left(\frac{e^{2z}}{4} + 1 \right) u_{yy} - \log(x^2 + y^2 + z^2)\, u = 0.$

c) $u_{xx} + 4u_{xy} + 3u_{yy} + 3u_x - u_y + 2u = 0.$

d) $a\,u_{xx} + 2a\,u_{xy} + a\,u_{yy} + b\,u_x + c\,u_y = 0, \qquad a \neq 0.$

Hint: Symmetric positive definite matrices have the property that all eigenvalues are strictly positive.

Question 3: [**Dimensionless form of PDEs, 4 Points**]
Consider the second order PDE

$$u_t - \varepsilon\Delta u + \mathbf{b} \cdot \nabla u + c\,u = f \quad \text{in } (0,T] \times \Omega,$$

known as the evolutionary convection–diffusion–reaction (CDR) equation. It models the transport of $u\,[\mathrm{mol}]$ in a fluid with diffusion coefficient $\varepsilon\,[\mathrm{m^2 s^{-1}}]$, convective field $\mathbf{b}\,[\mathrm{m\,s^{-1}}]$, reaction coefficient $c\,[\mathrm{s^{-1}}]$ and the source/sink term $f\,[\mathrm{mol\,s^{-1}}]$. Using a change of variables, write down the dimensionless form of the CDR equation.

Question 4: [**Consistency of Operators, 6 Points**]
Let the forward, backward, and central difference operators be denoted by Δ^+, Δ^-, and $\Delta^\pm$, respectively:

$$\Delta^+ u_i = \frac{u_{i+1} - u_i}{h}, \qquad \Delta^- u_i = \frac{u_i - u_{i-1}}{h}, \qquad \Delta^\pm u_i = \frac{u_{i+1} - u_{i-1}}{2h},$$

and let the second-order difference operator be denoted by δ^2:

$$\delta^2 u_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}.$$

Show that

a) $\Delta^\pm u_i = \frac{1}{2}\Big(\Delta^+ u_i + \Delta^- u_i\Big)$ and $\delta^2 u_i = \Delta^+\Big(\Delta^- u_i\Big)$.

b) For a sufficiently smooth function $v(x)$ evaluated at x_i show the estimates

$$\Delta^\pm v_i = v'(x_i) + \mathcal{O}(h^2), \qquad \delta^2 v_i = v''(x_i) + \mathcal{O}(h^2).$$

c) Determine the order of consistency of the approximation

$$u''(x) = \frac{1}{12h^2}\Big(- u(x+2h) + 16u(x+h) - 30u(x) + 16u(x-h) - u(x-2h)\Big).$$

Question 5: [**Five-Point Stencil, 6 Points**]
Consider the Poisson equation

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma, \end{aligned} \tag{1}$$

where Ω is a rectangular domain. Using a non-equidistant grid spacing as h_x and h_y in the x and y direction respectively, derive the five-point stencil

$$2\left(\theta^{-1}+\theta\right)\mathbf{u}_{i,j}-\theta^{-1}\left(\mathbf{u}_{i+1,j}+\mathbf{u}_{i-1,j}\right)-\theta\left(\mathbf{u}_{i,j+1}+\mathbf{u}_{i,j-1}\right)=h_x h_y \mathbf{f}_{i,j},$$

for the grid point (ih_x, jh_y) where $\theta = h_x/h_y$. Show that the local truncation error is given by

$$-\frac{1}{12}\left(h_x^2\partial_x^4 u + h_y^2\partial_y^4 u\right)\bigg|_{i,j} + \mathcal{O}(h^4),$$

where $h = \max\{h_x, h_y\}$.

Question 6: [**Non-Constant Diffusion, 6 Points**]
Consider the differential operator

$$\mathcal{L}u = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right), \tag{2}$$

where $k \in \mathcal{C}^4$ and the finite-difference approximation

$$\left(\mathcal{L}_h\mathbf{u}\right)_i = \frac{1}{h}\left(a_{i+1}\frac{\mathbf{u}_{i+1}-\mathbf{u}_i}{h} - a_i\frac{\mathbf{u}_i-\mathbf{u}_{i-1}}{h}\right), \tag{3}$$

on a uniform grid $x_i = x_0 + ih$ (assume u is sufficiently smooth). Here a_i is a grid function that approximates k at appropriate locations.
Show that the choices

a) $a_i = \dfrac{k_i + k_{i-1}}{2}$, and

b) $a_i = k\left(x_i - \frac{h}{2}\right)$,

each yield second-order consistency of $\mathcal{L}_h$ with $\mathcal{L}$ (i.e. the local truncation error is $\mathcal{O}(h^2)$).
Hints:

1. Use chain rule to write $\mathcal{L}u$ in terms of u_x and u_{xx}.
2. Use forward (Δ^+) and backward (Δ^-) difference approximation to write $\left(\mathcal{L}_h u\right)_i$ in terms of $u_x|_i$, $u_{xx}|_i$, and $u_{xxx}|_i$.
3. The local truncation error is given by $\mathcal{L}u|_i - \mathcal{L}_h u|_i$.
4. From point 3. you will get the consistency constraints

$$\frac{a_{i+1}-a_i}{h} = k'(x_i) + \mathcal{O}(h^2), \qquad \frac{a_{i+1}+a_i}{2} = k(x_i) + \mathcal{O}(h^2).$$

5. Substitute the value of a_i to get the answer. You might need to expand k_{i+1} and k_{i-1}.

Question 7: [Programming, 10+10 Points]

Consider the Poisson equation

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma, \end{aligned}$$

where $\Gamma = \partial\Omega$ denotes the boundary of the domain.
We have derived the five-point stencil for solving the above PDE in the case $\Omega = [0,1]^2$ with uniform grid spacing. In the attached Python script, you will find the code for this case.
The right-hand side f and the boundary condition g are chosen such that the exact solution is

$$u(x, y) = x^4 y^5 - 17 \sin(xy).$$

From theory, we expect a convergence rate of order 2, which can be verified numerically in the provided code.
For bonus points, extend the implementation as follows:

a) Modify the code so that the five-point stencil can be applied on a rectangular domain, e.g. $\Omega = [0, 2] \times [0, 1]$, using uniform grid spacing.

b) Modify the code to allow for non-uniform grid spacing with mesh sizes h_x and h_y in the x- and y-directions, respectively.

c) (**Double Bonus Points**) Extend the code to handle the case of an L-shaped domain, i.e.,

$$\Omega = [0,1]^2 \setminus \Big\{[0.5, 1] \times [0.5, 1]\Big\}.$$

For checking the solution in this case, you can use Example 2.17 from the lecture notes where f and g are chosen such that the exact solution is given by $u(x, y) = \sin(\pi x)\cos(\pi y)$.