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MA643 Numerical Analysis of PDEs

Summer Semester 2025

03.09.2025

Assigment 2

Assignment must be completed in groups of 2 members and submitted via Google Classroom by 19.09.2025 at 23:59. If the assignment is not prepared using LaTeX, a clear scanned copy of the handwritten work must be uploaded. Ensure that the names and enrollment numbers of all group members are clearly written on the submission. Late submissions will not be accepted.

Question 1: [FDM in Polar Coordinates, 6 Points]

Consider the Laplace equation

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, && (1)\\ u &= g \quad \text{on } \Gamma = \partial\Omega, && (2) \end{aligned}$$

where $\Omega = \{(x, y) \in \mathbb{R}^2 : \; x^2 + y^2 < 1\}$ is the unit disk, and f, g are given smooth functions.

a) Using polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, show that (1) can be written as

$$-\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\right) = f(r, \theta), \qquad (r, \theta) \in (0, 1) \times (0, 2\pi),$$

with boundary condition $u(1, \theta) = g(\theta)$, $\theta \in [0, 2\pi)$, where g is 2π–periodic in θ.

b) Construct a second-order finite-difference scheme for the operator

$$\mathcal{L}u := \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$

on a uniform polar mesh.

Question 2: [Neumann Boundary Conditions, 6 Points]

Consider the Laplace equation

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega = (0, 1)^2, && (3)\\ \partial_{\mathbf{n}} u &= g \quad \text{on } \Gamma = \partial\Omega, && (4) \end{aligned}$$

Derive the finite-difference approximation for the grid equations at the following boundary-related grid points:

a) Nodes on the right boundary: $i = M, \; j = 1, 2, \ldots, M - 1$.

b) Nodes on the bottom boundary: $j = 0, \; i = 1, 2, \ldots, M - 1$.

c) Corner grid points: $(M, 0)$ and (M, M).

Question 3: [**Robin Boundary Conditions, 6 Points**]

Suppose that the Neumann boundary condition in the previous problem is replaced by the Robin boundary condition

$$-u_x + \sigma u = g(y) \quad \text{on } x = 0,$$

where σ is a constant. Construct the finite difference approximation for the boundary grid points $i = 0$, $j = 1, 2, \ldots, M-1$.

Question 4: [**Normed Spaces, 4 Points**]

Let $X = \mathbb{R}^2$ be the vector space of ordered pairs $x = (\xi_1, \xi_2)$. Show that each of the following defines a norm on X:

a) $\|x\|_1 = |\xi_1| + |\xi_2|$,

b) $\|x\|_2 = \Big(|\xi_1|^2 + |\xi_2|^2\Big)^{1/2}$,

c) $\|x\|_\infty = \max\{|\xi_1|, |\xi_2|\}$.

Question 5: [**Inner Product Spaces, 4 Points**]

Let $\big(X, (\,\cdot\,,\,\cdot\,)\big)$ be an inner product space. Verify by direct calculations that for any $x, y, z \in X$ the following identity holds:

$$\|z - x\|^2 + \|z - y\|^2 = \tfrac{1}{2}\|x - y\|^2 + 2\left\|z - \tfrac{1}{2}(x + y)\right\|^2,$$

where $\|\,\cdot\,\| = \sqrt{(\,\cdot\,,\,\cdot\,)}$.

Question 6: [**Inner Product, 4 Points**]

Let $\mathrm{L}^2\Big((0,\infty), e^{-x}\,dx\Big)$ define the weighted space, i.e.,

$$\mathrm{L}^2\Big((0,\infty), e^{-x}\,dx\Big) = \left\{u : \int_0^\infty |u(x)|^2 e^{-x}\,dx < \infty\right\}.$$

Show that

$$a(u, v) = \int_0^\infty e^{-x}\,u(x)v(x)\,dx$$

defines an inner product on the weighted space.

Question 7: [**Programming, 10 Points**]

Consider the Poisson equation

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma, \end{aligned}$$

where $\Gamma = \partial\Omega$ denotes the boundary of the domain.
We have derived the nine-point stencil for solving the above PDE in the case $\Omega = [0,1]^2$ with uniform grid spacing.
The right-hand side f and the boundary condition g are chosen such that the exact solution is

$$u(x,y) = x^4 y^5 - 17\sin(xy).$$

From theory, we expect a convergence rate of order 4, which can be verified numerically.
For bonus points, extend the implementation of the five-point stencil to the nine-point stencil and verify the order of convergence.