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MA643 Numerical Analysis of PDEs
Summer Semester 2025
16.09.2025

Assigment 3

Assignment must be completed in groups of 2 members and submitted via Google Classroom by 17.10.2025 at 23:59. If the assignment is not prepared using LaTeX, a clear scanned copy of the handwritten work must be uploaded. Ensure that the names and enrollment numbers of all group members are clearly written on the submission. Late submissions will not be accepted.

Question 1: [$\|\cdot\|_p$ **Norm, 6 Points**]

a) Let $r \in [1,\infty)$, $p, q \in (1,\infty)$ with

$$\frac{1}{p} + \frac{1}{q} = 1,$$

and functions $u \in \mathrm{L}^{rp}(\Omega)$, $v \in \mathrm{L}^{rq}(\Omega)$. Show that

$$\|uv\|_{\mathrm{L}^r(\Omega)} \leq \|u\|_{\mathrm{L}^{rp}(\Omega)} \|v\|_{\mathrm{L}^{rq}(\Omega)}.$$

b) Let $f : (-1,1) \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} |x|^a, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

For what values of $a \in \mathbb{R}$ does $f \in \mathrm{L}^p((-1,1))$?

c) Let $B_1(\mathbf{0}) = \left\{ \boldsymbol{x} : \ \ \|\boldsymbol{x}\|_2 < 1 \right\}$ be the d-dimensional open unit ball centered at $\mathbf{0}$ for $d > 1$. Define

$$f(\boldsymbol{x}) = \begin{cases} \|\boldsymbol{x}\|_2^a, & \boldsymbol{x} \neq \mathbf{0}, \\ 0, & \boldsymbol{x} = \mathbf{0}, \end{cases}$$

Find the values of $a \in \mathbb{R}$ for which $f \in \mathrm{L}^p(B_1(\mathbf{0}))$ with $p \in [1,\infty)$.

Hint: In d-dimension the Jacobian for polar coordinate integral is r^{d-1}, for e.g., in 2D we have

$$\int\int f(x,y)dxdy = \int\int f(r,\theta) r dr \, d\theta.$$

Question 2: [**Weak Derivative, 6 Points**]

a) Let $f(x) = 1$ in Ω. Determine whether $f \in \mathrm{L}^1(\Omega)$ and $f \in \mathrm{L}^1_{\text{loc}}(\Omega)$ for

$$\Omega = (0,1) \quad \text{and} \quad \Omega = \mathbb{R}.$$

b) Show, using the definition of the weak derivative, that

$$D_{\mathrm{w}}^1 f(x) = \begin{cases} -2x, & -1 \le x < 0, \\ 0, & x = 0, \\ 2x, & 1 \ge x > 0, \end{cases} \qquad D_{\mathrm{w}}^2 f(x) = \begin{cases} -2, & -1 \le x < 0, \\ 0, & x = 0, \\ 2, & 1 \ge x > 0, \end{cases}$$

are the first and second weak derivatives of

$$f(x) = x|x|,$$

defined over $[-1, 1]$.

Question 3: [**Sobolev Space, 4 Points**]
Let Ω be a Lipschitz bounded domain and $1 \le p < \infty$. Show that the mapping defined by

$$u \mapsto \left(\int_\Omega |\nabla u|^p + \int_\Gamma |\mathrm{tr}(u)|^p \right)^{1/p}$$

is a norm over $\mathrm{W}^{1,p}(\Omega)$, where tr is the trace of u.
Hint: The Minkowski inequality for summation is given by

$$\left(\sum_{i=1}^n |x_i + y_i|^p \right)^{1/p} \le \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} + \left(\sum_{i=1}^n |y_i|^p \right)^{1/p}.$$

Question 4: [**Variational Problem, 4 Points**]
Let V be the space of continuous functions on $[0, 1]$ whose derivatives are piecewise continuous and bounded on $[0, 1]$, and that satisfy $v(0) = v(1) = 0$. Show that the problem of finding $u \in V$ such that

$$(u', v') = (f, v) \quad \forall v \in V$$

has a unique solution. Here $(\,\cdot\,,\,\cdot\,)$ denotes the L^2 inner product and $f \in \mathcal{C}[0, 1]$.

Question 5: [**Coercitivity, 4 Points**]
Consider the differential equation

$$-u'' + ku' + u = f.$$

Find a value of k such that there exists a non-trivial function $v \in \mathrm{H}^1(0, 1)$ with

$$a(v, v) = 0 \quad \text{but} \quad v \not\equiv 0.$$

Question 6: [**Variational Problem, 6 Points**]
Consider the fourth-order boundary value problem on the interval $[0, 1]$:

$$u^{(4)}(x) = f(x), \quad x \in (0, 1),$$

with homogeneous boundary conditions

$$u(0) = u(1) = u'(0) = u'(1) = 0.$$

a) Derive the variational formulation of this problem.

b) In which function space does the weak solution u belong?