MA643 Numerical Analysis of PDEs

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Assigment 4

Assignment must be completed in groups of 2 members and submitted via Google Classroom by 15.11.2025 at 23:59. If the assignment is not prepared using LaTeX, a clear scanned copy of the handwritten work must be uploaded. Ensure that the names and enrollment numbers of all group members are clearly written on the submission. Late submissions will not be accepted.

Question 1: [**Unisolvence, 4 Points**]

Let $\mathcal{P}_K$ be a finite-dimensional vector space over $\mathbb{R}$ and let

$$\Sigma_K = \{\Phi_{K,i}\}_{i=1}^{N_K}$$

be a collection of linear nodal functionals on $\mathcal{P}_K$. Assume $N_K = \dim \mathcal{P}_K$ and suppose there exists a local basis $\{\varphi_i\}_{i=1}^{N_K} \subset \mathcal{P}_K$ satisfying

$$\Phi_{K,i}(\varphi_j) = \delta_{ij} \qquad (1 \le i, j \le N_K).$$

Show that Σ_K is unisolvent for $\mathcal{P}_K$.

Question 2: [**Rectangular Elements, 3+3 Points**]

Let K be a quadrilateral in $\mathbb{R}^2$ with vertices $\{\mathbf{a}_i\}_{i=0}^{3}$ ordered counter–clockwise and with $\mathbf{a}_0$ the bottom–left vertex. Let the reference quadrilateral be $\hat{K} = [-1,1] \times [-1,1]$ with corner numbering

$$\hat{\mathbf{a}}_0 = (-1,-1), \quad \hat{\mathbf{a}}_1 = (1,-1), \quad \hat{\mathbf{a}}_2 = (1,1), \quad \hat{\mathbf{a}}_3 = (-1,1).$$

Let $F_K : \hat{K} \to K$ be an affine bijective map.

a) Compute the local basis functions $\{\hat{\varphi}_i\}_{i=0}^{3}$ on $\hat{K}$ (the usual $\mathbb{Q}_1$ basis on the square).

b) Give an explicit representation of the map F_K. State the necessary and sufficient condition on the physical vertices $\{\mathbf{a}_i\}$ for F_K to be affine.

Question 3: [**Robin Boundary Value Problem, 6 Points**]

Consider the Robin boundary value problem

$$-\Delta u = f \quad \text{in } \Omega,$$

$$\gamma\, u + \frac{\partial u}{\partial \mathbf{n}} = g \quad \text{on } \Gamma,$$

where $\gamma \in \mathbb{R}$ is a constant, $\Omega \subset \mathbb{R}^d$ $(d \ge 1)$ is a bounded Lipschitz domain with boundary $\Gamma = \partial\Omega$, and

$$f \in \mathrm{L}^2(\Omega), \qquad g \in \mathrm{L}^2(\Gamma).$$

a) Give a variational formulation of the problem.

b) Under what conditions does the bilinear form $a(\,\cdot\,,\,\cdot\,)$ satisfy symmetry, continuity, and coercivity required for application of the Lax–Milgram theorem?

Question 4: [Bilinear Finite Element, 6 Points]

Let $\mathcal{P}_K = \mathbb{Q}_1$, i.e., the space of bilinear finite element functions on a rectangular element K. Show that if the nodal values are chosen at the midpoints of the four edges of K, as illustrated in Fig. 1, then these nodal values do not uniquely determine a function in $\mathcal{P}_K$.

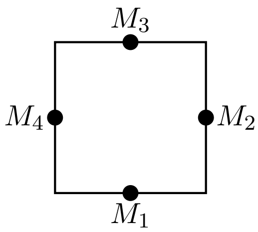

Abbildung 1: Edge midpoint nodes for the bilinear element.

Question 5: [Crouzeix-Raviart Element, 4 Points]

Let $\mathbb{P}_1^{\mathrm{nc}}(\mathcal{T}_h)$ be the Crouzeix–Raviart finite element space on a triangulation $\mathcal{T}_h$ of a connected polygonal domain $\Omega \subset \mathbb{R}^2$, with the additional condition that functions in the space vanish at the midpoints of boundary edges. Show that

$$\|v_h\|_h := \Big(\sum_{K\in\mathcal{T}_h} \|\nabla v_h\|^2_{L^2(K)} \Big)^{1/2}$$

defines a norm on $\mathbb{P}_1^{\mathrm{nc}}(\mathcal{T}_h)$.

Question 6: [Implementation of FEM, 4 Points]

Consider the Poisson problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma, \end{aligned}$$

where $\Omega = [0,2]^2$. The domain is triangulated using a criss–cross mesh with vertices

$$(0,0),\ (2,0),\ (2,2),\ (0,2),\ \text{and the interior node } (1,1).$$

The triangulation is obtained by connecting the interior node $(1,1)$ to each of the four corner vertices, thus forming four triangles. Using conforming piecewise linear finite elements ($\mathbb{P}_1$ elements), derive the finite element system of equations

$$\mathbf{A}\mathbf{u} = \mathbf{b}.$$

a) Write down the local stiffness matrices for each triangle.

b) Assemble the global stiffness matrix $\mathbf{A}$ and the load vector $\mathbf{b}$ corresponding to this triangulation.

c) Indicate how Dirichlet boundary conditions at the four corner nodes are incorporated into the system.

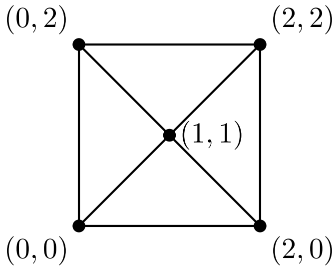

Abbildung 2: Criss–cross triangulation of $\Omega = [0, 2]^2$ for the $\mathbb{P}_1$ finite element.

Question 7: [Programming, 10 Points]

Consider the domain $\Omega = [0, 1]^2$. Write a Python code to generate a triangulation of the domain.

The code should take the parameter n_x, which divides each edge into $n_x + 1$ equally spaced points, and return:

- the Cartesian coordinates of all vertices, and
- the cell connectivity (vertex indices of each triangular cell stored in counter-clockwise order).

Additionally, the code should include an input parameter that specifies the meshing pattern as follows:

a) `up`: each square is divided by the diagonal from $(0, 0)$ to $(1, 1)$;

b) `down`: each square is divided by the diagonal from $(0, 1)$ to $(1, 0)$.

Further, extend the code to include an optional Boolean parameter `unstruct`. If `unstruct = True`, the function should generate an *unstructured triangulation* of the domain (for example, by randomly choosing the diagonal in each square or using Delaunay triangulation).

The output should ensure that all cell vertex coordinates are stored in counter-clockwise orientation.