A Residual based a Posteriori Error Estimators for Algebraic Flux Correction Scheme

Abhinav Jha

Applied and Computational Mathematics, RWTH Aachen University

Bound-Preserving Space and Time Discretizations for Convection-Dominated Problems
25th August 2021

Outline

- Steady-state convection-diffusion-reaction equation

$$\begin{aligned} -\varepsilon\Delta u + \mathbf{b}\cdot\nabla u + cu &= f && \text{in } \Omega \\ u &= u_b && \text{on } \Gamma_{\mathrm{D}}, \\ -\varepsilon\nabla u\cdot\mathbf{n} &= g && \text{on } \Gamma_{\mathrm{N}} \end{aligned} \tag{1}$$

  - Ω – bounded polyhedral Lipschitz domain in $\mathbb{R}^d$, $d \in \{2,3\}$
  - $\mathbf{n}$ – outward pointing unit normal
  - Assume

$$\left(c(x) - \frac{1}{2}\nabla\cdot\mathbf{b}(x)\right) \geq \sigma > 0$$

  - Interested in convection-dominated regime, $\varepsilon \ll \|\mathbf{b}\|_{L^\infty(\Omega)}$

- Ideal discretization
  1. Accurate and sharp layers
     - Many discretizations satisfy this property, e.g., SUPG
     - Reasonably well for AFC schemes

[1] J.,John: CAMWA (78), 3117-3138, 2019

- Ideal discretization
  1. Accurate and sharp layers
     - Many discretizations satisfy this property, e.g., SUPG
     - Reasonably well for AFC schemes
  2. Physically consistent results (no spurious oscillations)
     - Most discretizations violate this property, e.g., SUPG, SOLD schemes
     - Satisfied for AFC schemes

[1] J.,John: CAMWA (78), 3117-3138, 2019

- Ideal discretization
  1. Accurate and sharp layers
     - Many discretizations satisfy this property, e.g., SUPG
     - Reasonably well for AFC schemes
  2. Physically consistent results (no spurious oscillations)
     - Most discretizations violate this property, e.g., SUPG, SOLD schemes
     - Satisfied for AFC schemes
  3. Efficient computation of the solutions
     - Satisfied for linear discretizations
     - Usually not satisfied for nonlinear discretizations, like AFC schemes [1]

[1] J.,John: CAMWA (78), 3117-3138, 2019

- Ideal discretization
  1. Accurate and sharp layers
     - Many discretizations satisfy this property, e.g., SUPG
     - Reasonably well for AFC schemes
  2. Physically consistent results (no spurious oscillations)
     - Most discretizations violate this property, e.g., SUPG, SOLD schemes
     - Satisfied for AFC schemes
  3. Efficient computation of the solutions
     - Satisfied for linear discretizations
     - Usually not satisfied for nonlinear discretizations, like AFC schemes [1]
- Because of 2^{nd} property: AFC schemes very well suited for applications

[1] J.,John: CAMWA (78), 3117-3138, 2019

- Variational problem for AFC scheme
  Find $u_h \in V_h$ such that

$$a_h(u_h, v_h) + d_h(u_h; u_h, v_h) = \langle f, v_h \rangle \quad \forall\, v_h \in V_h$$

  - $V_h -$ finite element space with homogeneous Dirichlet boundary conditions $(V_h \subset V)$
  - stabilization

$$d_h(w; z, v) = \sum_{i,j=1}^{N} (1 - \alpha_{ij}(w)) d_{ij}(z_j - z_i) v_i \quad \forall\, w, v, z \in V_h$$

[1] Barrenechea, John, Knobloch, Rankin: SeMA Journal (75), 655–685, 2018

- Variational problem for AFC scheme
  Find $u_h \in V_h$ such that

$$a_h(u_h, v_h) + d_h(u_h; u_h, v_h) = \langle f, v_h \rangle \quad \forall\, v_h \in V_h$$

  - $V_h -$ finite element space with homogeneous Dirichlet boundary conditions $(V_h \subset V)$
  - stabilization

$$d_h(w; z, v) = \sum_{i,j=1}^{N} (1 - \alpha_{ij}(w)) d_{ij} (z_j - z_i) v_i \quad \forall\, w, v, z \in V_h$$

- Another representation of stabilization for $w, v, z \in V_h$,[1]

$$d_h(w; z, v) = \sum_{E \in \mathcal{E}_h} (1 - \alpha_E(w)) d_E h_E \left(\nabla z \cdot \mathbf{t}_E, \nabla v \cdot \mathbf{t}_E \right)$$

[1] Barrenechea, John, Knobloch, Rankin: SeMA Journal (75), 655–685, 2018

- AFC norm

$$\|u_h\|_{\mathrm{AFC}}^2 = \|u_h\|_a^2 + d_h(u_h, u_h, u_h) \quad \forall u_h \in V_h$$

  - where $\|u_h\|_a^2 = \varepsilon |u_h|_1^2 + \sigma \|u_h\|_0^2$

[1] John, Novo: CMAME (255), 289–305, 2013

Residual Based Approach

- AFC norm

$$\|u_h\|_{\mathrm{AFC}}^2 = \|u_h\|_a^2 + d_h(u_h, u_h, u_h) \quad \forall u_h \in V_h$$

  - where $\|u_h\|_a^2 = \varepsilon |u_h|_1^2 + \sigma \|u_h\|_0^2$

- Let $I_h u$ denote the Scott-Zhang interpolation operator. Galerkin orthogonality arguments

$$\begin{aligned} \|u - u_h\|_{\mathrm{AFC}}^2 &= \langle f, u - I_h u\rangle + \langle g, u - I_h u\rangle_{\Gamma_N} - a_h(u_h, u - I_h u) \\ &\quad + d_h(u_h; u, I_h u - u_h) \end{aligned}$$

[1] John, Novo: CMAME (255), 289–305, 2013

- AFC norm

$$\|u_h\|_{\mathrm{AFC}}^2 = \|u_h\|_a^2 + d_h(u_h, u_h, u_h) \quad \forall u_h \in V_h$$

  - where $\|u_h\|_a^2 = \varepsilon |u_h|_1^2 + \sigma \|u_h\|_0^2$
- Let $I_h u$ denote the Scott-Zhang interpolation operator. Galerkin orthogonality arguments

$$\begin{aligned}\|u - u_h\|_{\mathrm{AFC}}^2 &= \langle f, u - I_h u\rangle + \langle g, u - I_h u\rangle_{\Gamma_N} - a_h(u_h, u - I_h u) \\ &\quad + d_h(u_h; u, I_h u - u_h)\end{aligned}$$

- Standard residual a posteriori error bound [1]

$$\begin{aligned}&\langle f, u - I_h u\rangle + \langle g, u - I_h u\rangle_{\Gamma_N} - a_h(u_h, u - I_h u) \\ &= \sum_{K \in \mathcal{T}_h} (R_K(u_h), u - I_h u)_K + \sum_{F \in \mathcal{F}_h} \langle R_F(u_h), u - I_h u\rangle_F\end{aligned}$$

[1] John, Novo: CMAME (255), 289–305, 2013

with

$$R_K(u_h) \quad := \quad f + \varepsilon \Delta u_h - \mathbf{b} \cdot \nabla u_h - c u_h|_K,$$

$$R_F(u_h) \quad := \quad \begin{cases} -\varepsilon [|\nabla u_h \cdot \mathbf{n}_F|]_F & \text{if } F \in \mathcal{F}_{h,\Omega}, \\ g - \varepsilon (\nabla u_h \cdot \mathbf{n}_F) & \text{if } F \in \mathcal{F}_{h,N}, \\ 0 & \text{if } F \in \mathcal{F}_{h,D} \end{cases}$$

with

$$R_K(u_h) := f + \varepsilon \Delta u_h - \mathbf{b} \cdot \nabla u_h - c u_h|_K,$$

$$R_F(u_h) := \begin{cases} -\varepsilon [|\nabla u_h \cdot \mathbf{n}_F|]_F & \text{if } F \in \mathcal{F}_{h,\Omega}, \\ g - \varepsilon(\nabla u_h \cdot \mathbf{n}_F) & \text{if } F \in \mathcal{F}_{h,N}, \\ 0 & \text{if } F \in \mathcal{F}_{h,D} \end{cases}$$

- Using interpolation estimates, Cauchy-Schwarz, and Young's inequality

$$\begin{aligned} &\|u - u_h\|_a^2 + \frac{C_Y}{C_Y - 1} d_h(u_h; u - u_h, u - u_h) \\ &\leq \frac{C_Y^2}{2(C_Y - 1)} \sum_{K \in \mathcal{T}_h} \min\left\{ \frac{C_I^2}{\sigma}, \frac{C_I^2 h_K^2}{\varepsilon} \right\} \|R_K(u_h)\|_{L^2(K)}^2 \\ &\quad + \frac{C_Y^2}{2(C_Y - 1)} \sum_{F \in \mathcal{F}_h} \min\left\{ \frac{C_F^2 h_F}{\varepsilon}, \frac{C_F^2}{\sigma^{1/2} \varepsilon^{1/2}} \right\} \|R_F(u_h)\|_{L^2(F)}^2 \\ &\quad + \frac{C_Y}{C_Y - 1} d_h(u_h; u, I_h u - u_h) \end{aligned}$$

- Linearity of $d_h(\cdot;\cdot,\cdot)$,

$$d_h(u_h;u,I_hu-u_h) = d_h(u_h;u-u_h,I_hu-u_h) + d_h(u_h;u_h,I_hu-u_h)$$

- Linearity of $d_h(\cdot;\cdot,\cdot)$,

$$d_h(u_h; u, I_h u - u_h) = d_h(u_h; u - u_h, I_h u - u_h) + d_h(u_h; u_h, I_h u - u_h)$$

- Using interpolation estimates, Cauchy-Schwarz, trace inequality, inverse estimate, and Young's inequality

$$\begin{aligned} d_h(u_h; u_h, I_h u - u_h) \quad &\leq \quad \frac{C_Y}{2} \sum_{E \in \mathcal{E}_h} \min\left\{ \frac{\kappa_1 h_E^2}{\varepsilon}, \frac{\kappa_2}{\sigma} \right\} (1 - \alpha_E)^2 |d_E|^2 h_E^{1-d} \\ &\quad \times \|\nabla u_h \cdot \mathbf{t}_E\|_{L^2(E)}^2 + \frac{1}{C_Y} \|u - u_h\|_a^2, \end{aligned}$$

where

$$\begin{aligned} \kappa_1 &= C_{\text{edge,max}} \left(1 + (1 + C_I)^2\right), \\ \kappa_2 &= C_{\text{inv}}^2 C_{\text{edge,max}} \left(1 + (1 + C_I)^2\right). \end{aligned}$$

Theorem (Global a posteriori error estimate)

A global a posteriori error estimate for the energy norm is given by

$$\|u - u_h\|_a^2 \leq \eta_1^2 + \eta_2^2 + \eta_3^2,$$

where

$$\begin{aligned}
\eta_1^2 &= \sum_{K\in\mathcal{T}_h} \min\left\{\frac{4C_I^2}{\sigma}, \frac{4C_I^2 h_K^2}{\varepsilon}\right\} \|R_K(u_h)\|_{L^2(K)}^2, \\
\eta_2^2 &= \sum_{F\in\mathcal{F}_h} \min\left\{\frac{4C_F^2 h_F}{\varepsilon}, \frac{4C_F^2}{\sigma^{1/2}\varepsilon^{1/2}}\right\} \|R_F(u_h)\|_{L^2(F)}^2, \\
\eta_3^2 &= \sum_{E\in\mathcal{E}_h} \min\left\{\frac{4\kappa_1 h_E^2}{\varepsilon}, \frac{4\kappa_2}{\sigma}\right\} (1-\alpha_E)^2 |d_E|^2 h_E^{1-d} \|\nabla u_h \cdot \mathbf{t}_E\|_{L^2(E)}^2
\end{aligned}$$

- Formal local lower bound for a mesh cell K

$$\eta_K^2 = \eta_{\mathrm{Int,K}}^2 + \sum_{F \in \mathcal{F}_h(K)} \eta_{\mathrm{Face},F}^2 + \sum_{E \in \mathcal{E}_h(K)} \eta_{d_h,E}^2$$

- Formal local lower bound for a mesh cell K

$$\eta_K^2 = \eta_{\mathrm{Int,K}}^2 + \sum_{F\in\mathcal{F}_h(K)} \eta_{\mathrm{Face},F}^2 + \sum_{E\in\mathcal{E}_h(K)} \eta_{d_h,E}^2$$

where

$$\eta_{\mathrm{Int},K}^2 = \min\left\{\frac{4C_I^2}{\sigma}, \frac{4C_I^2 h_K^2}{\varepsilon}\right\} \|R_{K,h}(u_h)\|_{L^2(K)}^2,$$

$$\eta_{\mathrm{Face},F}^2 = \frac{1}{2}\min\left\{\frac{4C_F^2 h_F}{\varepsilon}, \frac{4C_F^2}{\sigma^{1/2}\varepsilon^{1/2}}\right\} \|R_F(u_h)\|_{L^2(F)}^2,$$

$$\eta_{d_h,E}^2 = \min\left\{\frac{4\kappa_1 h_E^2}{\varepsilon}, \frac{4\kappa_2}{\sigma}\right\}(1-\alpha_E)^2|d_E|^2 h_E^{1-d}\|\nabla u_h\cdot \mathbf{t}_E\|_{L^2(E)}^2$$

- Using standard bubble function arguments

$$
\begin{aligned}
\eta_{\mathrm{Int},K} \quad \leq \quad & C\Bigg(\max\left\{C_K^2 + \frac{C_K h_K}{\varepsilon}\|\mathbf{b}\|_{L^\infty(K)}, \frac{C_K}{\sigma}\|c\|_{L^\infty(K)}\right\}\|u-u_h\|_{a,K} \\
& + \frac{h_K}{\varepsilon^{1/2}} C_K\Big(\|f-f_h\|_{0,K} + \|(\mathbf{b}-\mathbf{b}_h)\cdot\nabla u_h\|_{0,K} + \|(c-c_h)u_h\|_{0,K}\Big)\Bigg)
\end{aligned}
$$

- Using standard bubble function arguments

$$
\begin{aligned}
\eta_{\mathrm{Int},K} \quad \leq \quad & C\Bigg(\max\left\{C_K^2 + \frac{C_K h_K}{\varepsilon}\|\mathbf{b}\|_{L^\infty(K)}, \frac{C_K}{\sigma}\|c\|_{L^\infty(K)}\right\}\|u - u_h\|_{a,K} \\
& + \frac{h_K}{\varepsilon^{1/2}} C_K \left(\|f - f_h\|_{0,K} + \|(\mathbf{b} - \mathbf{b}_h)\cdot\nabla u_h\|_{0,K} + \|(c - c_h)u_h\|_{0,K}\right)\Bigg)
\end{aligned}
$$

and

$$
\begin{aligned}
\eta_{\mathrm{Face},F} \quad \leq \quad & C\Bigg(\max\left\{C_F + \frac{C_F h_F\|\mathbf{b}\|_{L^\infty(\omega_F)}}{\varepsilon}, \frac{C_F h_F\|c\|_{L^\infty(\omega_F)}}{\varepsilon^{1/2}\sigma^{1/2}}\right\}\|u - u_h\|_{a,\omega_F} \\
& + \delta_{F\in\mathcal{F}_{h,N}} \frac{h_F^{1/2}}{\varepsilon^{1/2}}\|g - g_h\|_{L^2(F)} \\
& + \sum_{K\in\omega_F}\Bigg[\eta_{\mathrm{Int},K} + \frac{h_K}{\varepsilon^{1/2}}\Big(\|f - f_h\|_{0,K} + \|(\mathbf{b} - \mathbf{b}_h)\cdot\nabla u_h\|_{0,K} \\
& + \|(c - c_h)u_h\|_{0,K}\Big)\Bigg]\Bigg)
\end{aligned}
$$

- For the stabilization term, from [1] we get

$$|d_E| \leq C\left(\varepsilon + \|\mathbf{b}\|_{L^\infty(\Omega)} h + \|c\|_{L^\infty(\Omega)} h^2\right) h_E^{d-2}$$

[1] Barrenechea, John, Knobloch, Rankin: SeMA Journal (75), 655–685, 2018

- For the stabilization term, from [1] we get

$$|d_E| \leq C\left(\varepsilon + \|\mathbf{b}\|_{L^\infty(\Omega)} h + \|c\|_{L^\infty(\Omega)} h^2\right) h_E^{d-2}$$

Hence,

$$\begin{aligned} \eta_{d_h,E} \quad \leq \quad & C \sum_{E \in \mathcal{E}_h} (1-\alpha_E)\left(\varepsilon + \|\mathbf{b}\|_{L^\infty(\Omega)} h + \|c\|_{L^\infty(\Omega)} h^2\right) \\ & \times \frac{h_E^{(3-d)/2}}{\varepsilon^{1/2}} \|\nabla u_h \cdot \mathbf{t}_E\|_{L^2(E)} \end{aligned}$$

[1] Barrenechea, John, Knobloch, Rankin: SeMA Journal (75), 655–685, 2018

Theorem (Formal local lower bound)

There exists a constant $C > 0$, independent of the size of elements of $\mathcal{T}$, such that, for every $K \in \mathcal{T}$, the following formal local lower bound holds

$$\eta_{\mathrm{Int},K} + \sum_{K \in \mathcal{F}_h(K)} \eta_{\mathrm{Face},F} + \sum_{E \in \mathcal{E}_h(K)} \eta_{d_h,E}$$

$$\begin{aligned}
\leq \quad & \max\left\{ C_K^2 + \frac{C_K h_K}{\varepsilon}\|\mathbf{b}\|_{L^\infty(K)}, \frac{C_K}{\sigma}\|c\|_{L^\infty(K)} \right\} \|u - u_h\|_{a,\omega_K} \\
& + C \sum_{K \in \omega_K} \frac{h_K}{\varepsilon^{1/2}} \left(\|f - f_h\|_{0,K} + \|(\mathbf{b} - \mathbf{b}_h)\cdot \nabla u_h\|_{0,K} + \|(c - c_h)u_h\|_{0,K} \right) \\
& + C \sum_{F \in \mathcal{F}_h(K)} \delta_{F \in \mathcal{F}_{h,N}} \frac{h_F^{1/2}}{\varepsilon^{1/2}} \|g - g_h\|_{L^2(F)} \\
& + \sum_{E \in \mathcal{E}_h(K)} h^{1-d/2} \frac{h^{1/2}}{\varepsilon^{1/2}} \left(\varepsilon + \|\mathbf{b}\|_{L^\infty(\Omega)} h + \|c\|_{L^\infty(\Omega)} h^2 \right) \|\nabla u_h \cdot \mathbf{t}_E\|_{L^2(E)}.
\end{aligned}$$

- The initial solution for the nonlinear loop is the SUPG solution [1]
  - $u_{\mathrm{AFC}} :=$ AFC solution
  - $u_{\mathrm{SUPG}} :=$ SUPG solution
- By triangle inequality

$$
\begin{aligned}
\|u - u_{\mathrm{AFC}}\|_a^2 &\leq 2\left(\|u - u_{\mathrm{SUPG}}\|_a^2 + \|u_{\mathrm{SUPG}} - u_{\mathrm{AFC}}\|_a^2\right) \\
&\leq 2\left(\|u - u_{\mathrm{SUPG}}\|_{\mathrm{SUPG}}^2 + \|u_{\mathrm{SUPG}} - u_{\mathrm{AFC}}\|_a^2\right)
\end{aligned}
$$

[1] J.John: BAIL 2018 (135), 2020

[2] John, Novo: CMAME (255), 289-305, 2013

- The initial solution for the nonlinear loop is the SUPG solution [1]
  - $u_{\text{AFC}} :=$ AFC solution
  - $u_{\text{SUPG}} :=$ SUPG solution
- By triangle inequality

$$
\begin{aligned}
\|u - u_{\text{AFC}}\|_a^2 &\leq 2\left(\|u - u_{\text{SUPG}}\|_a^2 + \|u_{\text{SUPG}} - u_{\text{AFC}}\|_a^2\right) \\
&\leq 2\left(\|u - u_{\text{SUPG}}\|_{\text{SUPG}}^2 + \|u_{\text{SUPG}} - u_{\text{AFC}}\|_a^2\right)
\end{aligned}
$$

- Using estimators from [2]

$$
\|u - u_{\text{SUPG}}\|_{\text{SUPG}}^2 \leq \eta_{\text{SUPG}}^2
$$

[1] J.John: BAIL 2018 (135), 2020

[2] John, Novo: CMAME (255), 289-305, 2013

- The initial solution for the nonlinear loop is the SUPG solution [1]
  - $u_{\text{AFC}} :=$ AFC solution
  - $u_{\text{SUPG}} :=$ SUPG solution
- By triangle inequality

$$\begin{aligned} \|u - u_{\text{AFC}}\|_a^2 &\leq 2\left(\|u - u_{\text{SUPG}}\|_a^2 + \|u_{\text{SUPG}} - u_{\text{AFC}}\|_a^2\right) \\ &\leq 2\left(\|u - u_{\text{SUPG}}\|_{\text{SUPG}}^2 + \|u_{\text{SUPG}} - u_{\text{AFC}}\|_a^2\right) \end{aligned}$$

- Using estimators from [2]

$$\|u - u_{\text{SUPG}}\|_{\text{SUPG}}^2 \leq \eta_{\text{SUPG}}^2$$

  and denoting

$$\eta_{\text{AFC}-\text{SUPG}}^2 := \|u_{\text{SUPG}} - u_{\text{AFC}}\|_a^2$$

  $\Rightarrow$

$$\|u - u_h\|_a^2 \leq 2\left(\eta_{\text{SUPG}}^2 + \eta_{\text{AFC}-\text{SUPG}}^2\right)$$

[1] J.John: BAIL 2018 (135), 2020

[2] John, Novo: CMAME (255), 289-305, 2013

- Standard strategy for solving

$$\textbf{SOLVE} \rightarrow \textbf{ESTIMATE} \rightarrow \textbf{MARK} \rightarrow \textbf{REFINE}$$

- Effectivity index for the estimator

$$\eta_{\text{eff}} = \frac{\eta}{\|u - u_h\|_a}$$

[1] Kuzmin: in Proc. Int. Conf. Comput. Meth. for Coupled Problems in Science and Engineering, CIMNE, 2007

[2] Barrenechea, John, Knobloch: M3AS (27), 525–548, 2017

[3] Barrenechea, John, Knobloch, Rankin: SeMA Journal (75), 655–685, 2018

- Standard strategy for solving

SOLVE → ESTIMATE → MARK → REFINE

- Effectivity index for the estimator

$$\eta_{\text{eff}} = \frac{\eta}{\|u - u_h\|_a}$$

- Limiters
  - Monolithic upwind (MU) limiter[1]
  - Linearity preservation (LP) limiter[2,3]

[1] Kuzmin: in Proc. Int. Conf. Comput. Meth. for Coupled Problems in Science and Engineering, CIMNE, 2007

[2] Barrenechea, John, Knobloch: M3AS (27), 525–548, 2017

[3] Barrenechea, John, Knobloch, Rankin: SeMA Journal (75), 655–685, 2018

- Comparison of results:
  - On effectivity index (η_{eff})
  - Adaptive grid refinement
  - Behavior of η_{d_h}
  - Behavior of η_{SUPG} and $\eta_{\mathrm{AFC-SUPG}}$
  - Smearing of internal layer[1]

[1] John, Knobloch: CMAME (197), 1997–2014, 2008

- Iterative solver for AFC schemes
  - Matrix formulation of the AFC schemes[1,2]

$$AU + (I - \alpha)\, DU = F$$

  - Fixed point right-hand side

$$(A + D)\, U^{\nu+1} = F + \omega\alpha D U^{\nu},$$

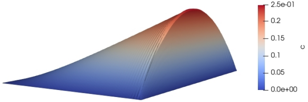
where $\omega > 0$ is a dynamic damping parameter

[1] J.John: BAIL 2018 (135), 2020

[2] J.,John: CAMWA (78), 3117-3138, 2019

- $\Omega = (0,1)^2$, $\varepsilon = 10^{-3}$, $\mathbf{b} = (2,1)^T$, $c = 1$ and f such that

$$u(x,y) = y(1-y)\left(x - \frac{e^{(x-1)/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}}\right)$$



- proposed in [1]
- $\mathbb{P}_1$ finite elements
- stop of the non linear iteration [2]
  - 25000 iterations
  - $\|\text{residual}\|_2 \leq \sqrt{\#\text{nDOFs}}10^{-10}$

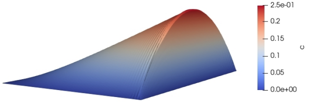
[1] Allendes et. al. : SISC 39(5):A1903-A1927, 2017

[2] J.,John: CAMWA (78), 3117-3138, 2019

Numerical Studies

- $\Omega = (0,1)^2, \varepsilon = 10^{-3}, \mathbf{b} = (2,1)^T, c = 1$ and f such that

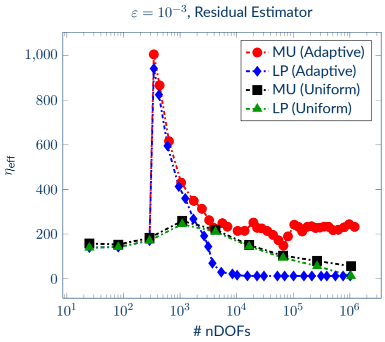
$$u(x,y) = y(1-y)\left(x - \frac{e^{(x-1)/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}}\right)$$



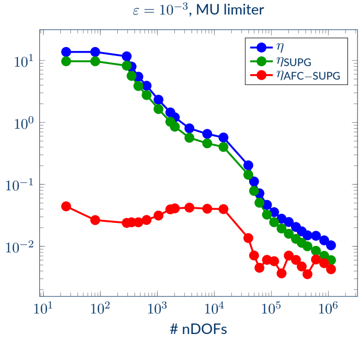
- proposed in [1]
- $\mathbb{P}_1$ finite elements
- stop of the non linear iteration [2]
  - 25000 iterations
  - $\|\text{residual}\|_2 \leq \sqrt{\#\text{nDOFs}}10^{-10}$
- stop of the adaptive algorithm
  - $\eta \leq 10^{-3}$
  - $\#\text{nDOFs} \approx 10^6$

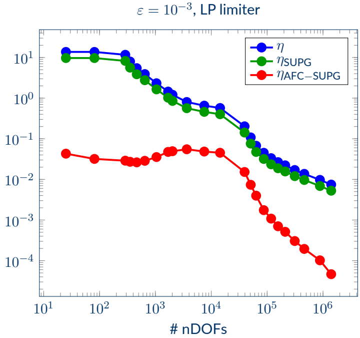
[1] Allendes et. al. : SISC 39(5):A1903-A1927, 2017
[2] J.,John: CAMWA (78), 3117-3138, 2019

- Effectivity index

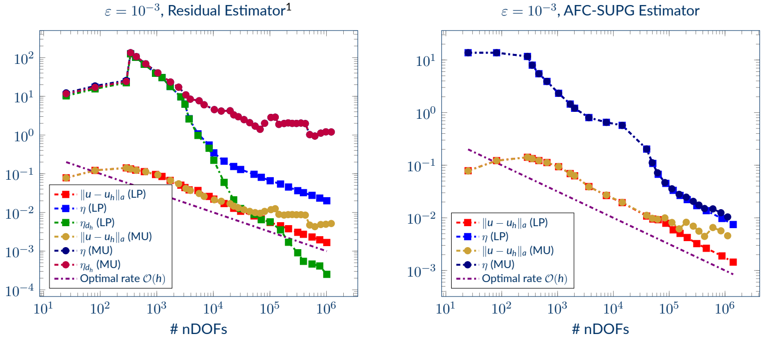


- Comparison of η_{SUPG} and $\eta_{\mathrm{AFC-SUPG}}$





- Errors on adaptive grids



[1] Barrenechea, John, Knobloch: SINUM (54), 2427–2451, 2016

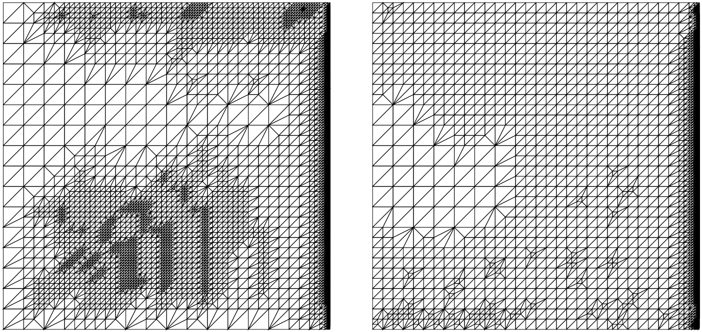
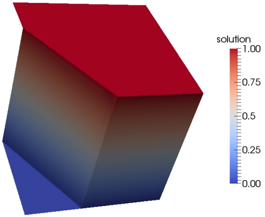


Figure 1: 14^{th} adaptively refined grid with residual estimator. MU limiter (#nDOFs $= 22962$) (left) and LP limiter (#nDOFs $= 23572$) (right)

- Example with interior and boundary layer[1]
- $\Omega = (0,1)^2, \varepsilon = 10^{-4}$, $\mathbf{b} = (\cos(-\pi/3), \sin(-\pi/3))^T$, $c = f = 0$

$$u_b = \begin{cases} 1 & (y = 1 \wedge x > 0) \text{ or } (x = 0 \wedge y > 0.7), \\ 0 & \text{else.} \end{cases}$$



[1] Hughes, Mallet, Mizukami: CMAME, 54(3), 341-345, 1986

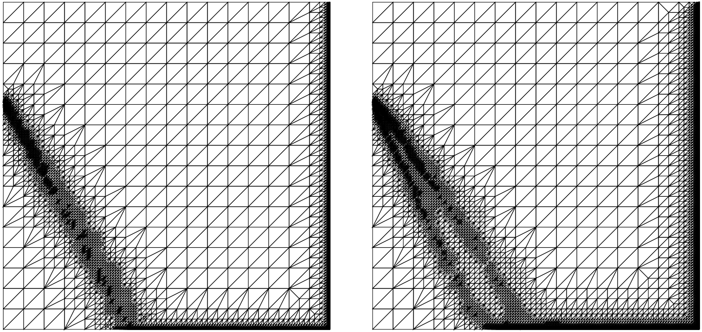
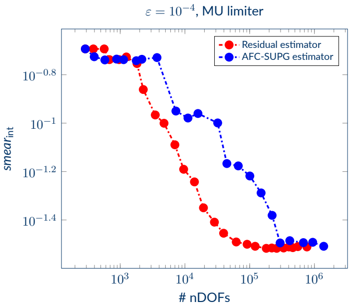
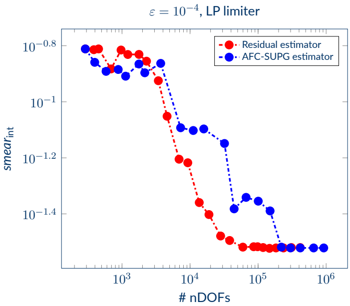


Figure 2: 14^{th} adaptively refined grid for MU limiter. Residual estimator (left) and AFC-SUPG estimator (right)

- Thickness of internal layer





- Hanging nodes
  - Preserves angles after red-refinement
  - Avoids prism and pyramids in 3D mesh refinement
  - *hp* adaptive refinement
- Certain stabilized schemes rely on the property of triangulation [1]

[1] Xu, Zikatanov: MC, 68(228), 1429–1446, 1999

Lemma

Let $\mathcal{T}$ be a non-conforming triangulation of Ω, i.e., $\mathcal{T}$ has hanging nodes. Then, for all $q \in H(\mathcal{T})$ there are coefficients a_{qp} with $p \in N_F(\mathcal{T}) \setminus H(\mathcal{T})$ such that all $v \in V_h$ can be represented as[1,2]

$$v(q) = \sum_{p \in N_F(\mathcal{T}) \setminus H(\mathcal{T})} a_{qp} v(p)$$

[1] Gräser : PhD Thesis, FU Berlin 2011

[2] Jha: PhD Thesis, FU Berlin 2020

Lemma

Let $\mathcal{T}$ be a non-conforming triangulation of Ω, i.e., $\mathcal{T}$ has hanging nodes. Then, for all $q \in H(\mathcal{T})$ there are coefficients a_{qp} with $p \in N_F(\mathcal{T}) \setminus H(\mathcal{T})$ such that all $v \in V_h$ can be represented as[1,2]

$$v(q) = \sum_{p \in N_F(\mathcal{T}) \setminus H(\mathcal{T})} a_{qp} v(p)$$

Theorem

Let $\{\mathcal{T}_0, \cdots, \mathcal{T}_j\}$ be a grid hierarchy on Ω with $\mathcal{T}_0$ being conforming. Let us denote $\mathcal{T} = \mathcal{T}_j$, i.e., the final refinement level. Then a basis of V_h is given by[1]

$$B(\mathcal{T}) := \left\{ \varphi_p = \varphi_p^{\mathrm{nc}} + \sum_{q \in H(\mathcal{T})} a_{qp} \varphi_q^{\mathrm{nc}} : p \in N_F(\mathcal{T}) \setminus H(\mathcal{T}) \right\}$$

[1] Gräser : PhD Thesis, FU Berlin 2011

[2] Jha: PhD Thesis, FU Berlin 2020

- Difficulties with monolithic upwind limiter
  - DMP is satisfied if[1]

$$a_{ij} + a_{ji} \leq 0, \tag{2}$$

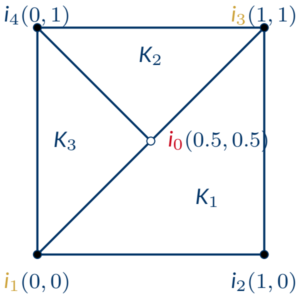
    where a_{ij} is in the stiffness matrix
  - Equivalent condition on conforming grids

$$\left(\nabla \varphi_i, \nabla \varphi_j\right) \leq 0$$

  - For grids with hanging nodes Eq. (2) fails

[1] Barrenechea, John, Knobloch: SINUM (54), 2427–2451, 2016

- Consider the convection-diffusion equation with $\varepsilon > 0$ and convective field $\mathbf{b} = (b_1, b_2)$



  - $$a_{13} + a_{31} = -\frac{1}{2}\varepsilon + \frac{(b_2 - b_1)}{6} > 0,$$

    for $b_1 = 0$ and $b_2 > 3\varepsilon$

- Modification for linearity preserving limiter
  - Assumptions for DMP preservation is satisfied
  - Modification required for the parameter γ_i[1]

$$\gamma_i = \frac{\max\limits_{x_j \in \partial \Delta_i} |x_i - x_j|}{\operatorname{dist}\left(x_i, \partial \Delta_i^{\text{conv}}\right)},$$

where $\Delta_i = \operatorname{supp}(\varphi_i)$

  - For computation of denominator replace Δ_i^{conv} with $\Delta_i^{T,\text{conv}}$ s.t.

$$\operatorname{dist}\left(x_i, \partial \Delta_i^{T,\text{conv}}\right) \leq \operatorname{dist}\left(x_i, \partial \Delta_i^{\text{conv}}\right)$$

[1] Barrenchea, John, Knobloch: M3AS (27), 525–548,2017

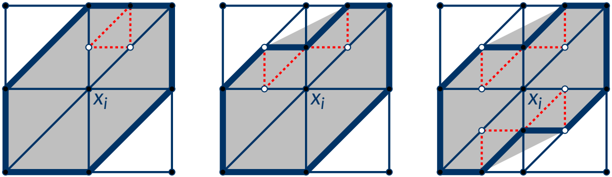


Figure 3: Examples of $\triangle_i$ for the node x_i with bold lines and $\triangle_i^{\text{conv}}$ with the shaded area

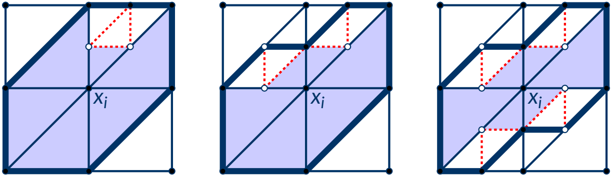


Figure 4: Examples of $\triangle_i^{T,\text{conv}}$ (blue) for the node x_i

Modified Monolithic Upwind Limiter

- Modified monolithic upwind (MMU) limiter[1]
  - Drops symmetric condition on α_{ij}
  - AFC system is modified

$$(A + D)\,U = F + (D - B)\,U,$$

where

$$b_{ij} = \max\left\{(1 - \overline{\alpha}_{ij}(u))\,a_{ij}, 0, (1 - \overline{\alpha}_{ji}(u))\,a_{ji}\right\}$$

  - $\{\overline{\alpha}_{ij}\}$ are computed similar to monolithic upwind limiter with

$$P_i^+ = \sum_{j=1, a_{ij}>0}^{N} a_{ij} \max\left\{(u_i - u_j), 0\right\}, \quad P_i^- = \sum_{j=1, a_{ij}>0}^{N} a_{ij} \min\left\{(u_i - u_j), 0\right\}$$

[1] Knobloch: ENUMATH Proc. 2019, 605–615, 2021

- Modified monolithic upwind (MMU) limiter[1]
  - Drops symmetric condition on α_{ij}
  - AFC system is modified

$$(A + D)\, U = F + (D - B)\, U,$$

where

$$b_{ij} = \max \left\{ \left(1 - \overline{\alpha}_{ij}(u)\right) a_{ij}, 0, \left(1 - \overline{\alpha}_{ji}(u)\right) a_{ji} \right\}$$

  - $\{\overline{\alpha}_{ij}\}$ are computed similar to monolithic upwind limiter with

$$P_i^+ = \sum_{j=1, a_{ij}>0}^{N} a_{ij} \max \left\{ (u_i - u_j), 0 \right\}, \quad P_i^- = \sum_{j=1, a_{ij}>0}^{N} a_{ij} \min \left\{ (u_i - u_j), 0 \right\}$$

  - Residual based estimator holds with

$$\eta_3^2 = \sum_{E \in \mathcal{E}_h} \min \left\{ \frac{4\kappa_1 h_E^2}{\varepsilon}, \frac{4\kappa_2}{\sigma} \right\} \left(|b_E - d_E|\right)^2 h_E^{1-d} \|\nabla u_h \cdot \mathbf{t}_E\|_{L^2(E)}^2$$

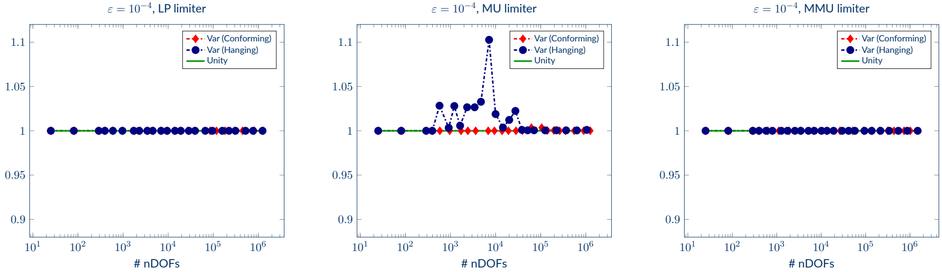
[1] Knobloch: ENUMATH Proc. 2019, 605–615, 2021

- Comparison of results on
  - Discrete maximum principle

$$\mathrm{Var}(u_h) = u_h^{\max} - u_h^{\min}$$

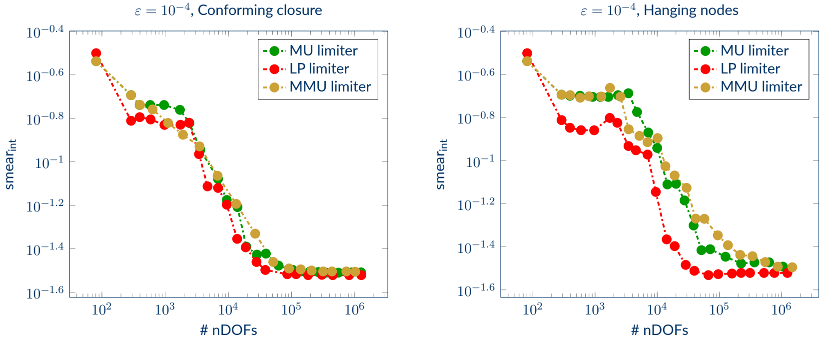
  - Smearing of internal layer
  - Error order

- Example with interior and boundary layer[1]



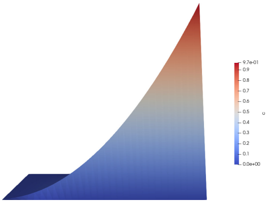
[1] Hughes, Mallet, Mizukami: CMAME, 54(3), 341-345, 1986

- Thickness of internal layer



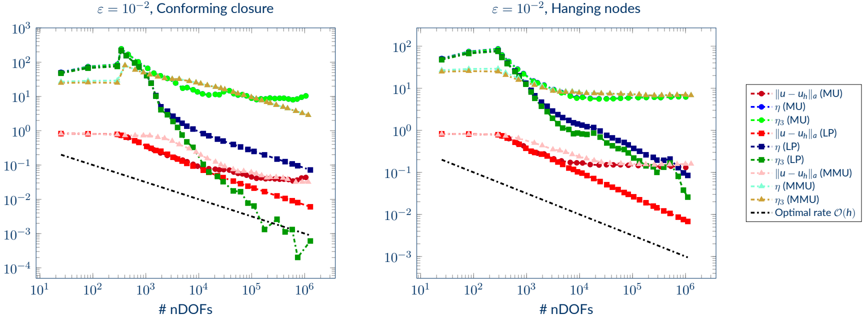
- Example with corner boundary layer[1]
- $\Omega = (0,1)^2, \varepsilon = 10^{-2}$, $\mathbf{b} = (2,3)^T$, $c = 1, u_b = 0$, and f such that

$$u(x,y) = xy^2 - y^2 \exp\left(\frac{2(x-1)}{\varepsilon}\right) - x\exp\left(\frac{3(y-1)}{\varepsilon}\right) + \exp\left(\frac{2(x-1)+3(y-1)}{\varepsilon}\right),$$



[1] John, Knobloch, Savescu: CMAME (200), 2916–2929, 2011

- Errors on adaptive grids



Conclusions and Outlook

- Conclusions[1]
  - Effectivity index not robust with residual based approach
  - For the AFC-SUPG estimator, the effectivity index was better
  - Choice of limiter did not play a role in AFC-SUPG estimator

[1] Jha: CAMWA, 97(1), 86–99, 2021

[2] J.,John: CAMWA (78), 3117-3138, 2019

- Conclusions[1]
  - Effectivity index not robust with residual based approach
  - For the AFC-SUPG estimator, the effectivity index was better
  - Choice of limiter did not play a role in AFC-SUPG estimator
  - For the MU limiter with the residual estimator reduced order of convergence
  - η_{d_h} is the dominating term in η for MU limiter if problem becomes locally diffusion-dominated. For LP limiter dominating term in the convection-dominated situation
  - With adaptive grid refinement, problem could become locally diffusion-dominated. Then use LP limiter
  - For a small diffusion coefficient, use MU limiter because nonlinear problems difficult to solve with LP limiter[2]
  - Residual estimator approximates the internal layer better

[1] Jha: CAMWA, 97(1), 86–99, 2021
[2] J.,John: CAMWA (78), 3117-3138, 2019

- Conclusions (contd.)
  - DMP violated for MU limiter on grids with conforming closure and on grids with hanging nodes
  - DMP satisfied for LP and MMU limiter irrespective of grid refinement
  - Same convergence for MU and MMU limiter
  - LP limiter gave the most satisfying results on grids with hanging nodes

- Outlook
  - Development of robust estimators
  - Numerical studies in 3D
  - Extending the analysis for the local efficiency of the estimator
  - Comparison with monolithic convex limiter[1]

[1] Kuzmin: CMAME (361), 112804, 2020