Recent Advancements in a Posteriori Error Estimation for Algebraic Stabilizations

Abhinav Jha & Naveed Ahmed

Indian Institute of Technology, Gandhinagar & GUST, Kuwait

ENUMATH 2025
2^{nd} September 2025

Outline

- Steady-state convection-diffusion-reaction equation

$$\begin{aligned} -\varepsilon\Delta u + \mathbf{b}\cdot\nabla u + cu &= f && \text{in } \Omega, \\ u &= u_{\mathrm{D}} && \text{on } \Gamma_{\mathrm{D}}, \\ -\varepsilon\nabla u\cdot\mathbf{n} &= u_{\mathrm{N}} && \text{on } \Gamma_{\mathrm{N}} \end{aligned}$$

  - Ω – bounded polyhedral Lipschitz domain in $\mathbb{R}^d$, $d \in \{2,3\}$
  - $\mathbf{n}$ – outward pointing unit normal
  - Assume

$$\left(c(x) - \frac{1}{2}\nabla\cdot\mathbf{b}(x)\right) \geq \sigma \geq 0$$

  - Interested in convection-dominated regime, $\varepsilon \ll \|\mathbf{b}\|_{L^\infty(\Omega)} L$
    - L – Characteristic length of the problem

- Ideal discretization
  1. Accurate and sharp layers [1]

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Algebraic Stabilization Schemes

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  3. Efficient computation of the solutions
- Because of 2^{nd} property: Algebraic stabilized schemes very well suited for applications
- Alternate approach: Adaptive grids
- Idea: Combine both the approaches

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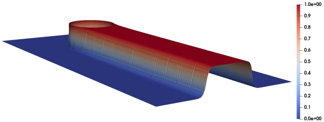
- Hemker Problem [1]: $\varepsilon = 10^{-4}$, $\mathbf{b} = (1,0)^\top$, $\#\text{dof} = 33496$



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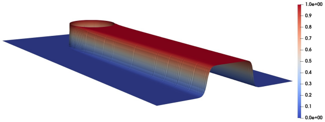

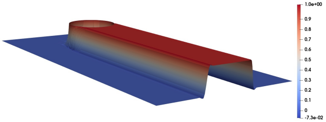

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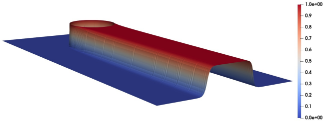

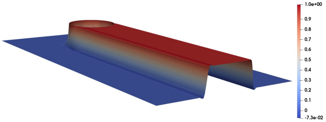

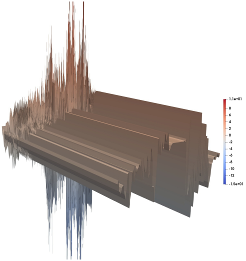

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- Fully Computable Error Bound
  - Implicit Residual Based Estimator[1]
  - Local Efficiency depends on Linearity Preserving

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[3] J.,John, Knobloch: *SIAM Journal on Scientific Computing*, 45 (**4**), B564-B589, 2023

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  - Implicit Residual Based Estimator[1]
  - Local Efficiency depends on Linearity Preserving
- Residual Based Error Estimator[2]
  - Explicit Residual Based Estimator
  - Formal Local Lower Bound
  - Hanging Nodes Theory[3]

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- Variational problem for AS scheme
  Find $u_h \in V_h$ such that

$$a_h(u_h, v_h) + b_h(u_h; u_h, v_h) = \langle f, v_h \rangle \quad \forall\, v_h \in V_h$$

  - V_h- finite element space with homogeneous Dirichlet boundary conditions $(V_h \subset V)$

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$$b_h(w; z, v) = \sum_{i,j=1}^{N} b_{ij}(w)(z_j - z_i)v_i \quad \forall\, w, z, v \in V_h$$

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- Another representation of stabilization for $w, v, z \in V_h$,[1]

$$b_h(w; z, v) = \sum_{E \in \mathcal{E}_h} |b_E(w)| h_E \left(\nabla z \cdot \mathbf{t}_E, \nabla v \cdot \mathbf{t}_E\right)$$

[1] Barrenechea, John, Knobloch, Rankin: *SeMA Journal*, 75 (**4**), 655–685, 2018

- AS norm

$$\|u_h\|_{\mathrm{As}}^2 = \|u_h\|_a^2 + b_h(u_h, u_h, u_h) \quad \forall u_h \in V_h$$

  - where $\|u_h\|_a^2 = \varepsilon|u_h|_1^2 + \sigma\|u_h\|_0^2$

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- Let $\mathcal{I}_h u$ denote the Scott-Zhang interpolation operator. Galerkin orthogonality arguments

$$\begin{aligned} \|u - u_h\|_{\mathrm{AS}}^2 &= \langle f, u - \mathcal{I}_h u\rangle + \langle u_{\mathrm{N}}, u - \mathcal{I}_h u\rangle_{\Gamma_{\mathrm{N}}} - a_h(u_h, u - \mathcal{I}_h u) \\ &\quad + b_h(u_h; u, \mathcal{I}_h u - u_h) \end{aligned}$$

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Residual Based Approach

- AS norm

$$\|u_h\|_{\mathrm{AS}}^2 = \|u_h\|_a^2 + b_h(u_h, u_h, u_h) \quad \forall u_h \in V_h$$

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- Standard residual a posteriori error bound [1]

$$\begin{aligned} &\langle f, u - \mathcal{I}_h u \rangle + \langle u_{\mathrm{N}}, u - \mathcal{I}_h u \rangle_{\Gamma_{\mathrm{N}}} - a_h(u_h, u - \mathcal{I}_h u) \\ &= \quad \sum_{K \in \mathcal{T}_h} (R_K(u_h), u - \mathcal{I}_h u)_K + \sum_{F \in \mathcal{F}_h} \langle R_F(u_h), u - \mathcal{I}_h u \rangle_F \end{aligned}$$

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with

$$
\begin{aligned}
R_K(u_h) &:= f + \varepsilon \Delta u_h - \mathbf{b} \cdot \nabla u_h - c u_h|_K, \\
R_F(u_h) &:= \begin{cases} -\varepsilon \left[\!\left[\nabla u_h \cdot \mathbf{n}_F \right]\!\right]_F & \text{if } F \in \mathcal{F}_{h,\Omega}, \\ u_{\mathrm{N}} - \varepsilon (\nabla u_h \cdot \mathbf{n}_F) & \text{if } F \in \mathcal{F}_{h,\mathrm{N}}, \\ 0 & \text{if } F \in \mathcal{F}_{h,\mathrm{D}} \end{cases}
\end{aligned}
$$

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- Using interpolation estimates, Cauchy-Schwarz, and Young's inequality

$$\begin{aligned} \|u - u_h\|_a^2 &+ \frac{C_Y}{C_Y - 1} b_h(u_h; u - u_h, u - u_h) \\ \leq &\frac{C_Y^2}{2(C_Y - 1)} \sum_{K \in \mathcal{T}_h} \min\left\{ \frac{C_{\mathcal{I}}^2}{\sigma}, \frac{C_{\mathcal{I}}^2 h_K^2}{\varepsilon} \right\} \|R_K(u_h)\|_{L^2(K)}^2 \\ &+ \frac{C_Y^2}{2(C_Y - 1)} \sum_{F \in \mathcal{F}_h} \min\left\{ \frac{C_F^2 h_F}{\varepsilon}, \frac{C_F^2}{\sigma^{1/2} \varepsilon^{1/2}} \right\} \|R_F(u_h)\|_{L^2(F)}^2 \\ &+ \frac{C_Y}{C_Y - 1} b_h(u_h; u, \mathcal{I}_h u - u_h) \end{aligned}$$

- Linearity of $b_h(\cdot;\cdot,\cdot)$,

$$b_h(u_h; u, \mathcal{I}_h u - u_h) = b_h(u_h; u - u_h, \mathcal{I}_h u - u_h) + b_h(u_h; u_h, \mathcal{I}_h u - u_h)$$

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- Using interpolation estimates, Cauchy-Schwarz, trace inequality, inverse estimate, and Young's inequality

$$\begin{aligned} b_h(u_h; u_h, \mathcal{I}_h u - u_h) \quad &\leq \quad \frac{C_Y}{2} \sum_{E \in \mathcal{E}_h} \min\left\{ \frac{\kappa_1 h_E^2}{\varepsilon}, \frac{\kappa_2}{\sigma} \right\} |b_E|^2 h_E^{1-d} \\ &\quad \times \|\nabla u_h \cdot \mathbf{t}_E\|_{L^2(E)}^2 + \frac{1}{C_Y} \|u - u_h\|_a^2, \end{aligned}$$

where

$$\begin{aligned} \kappa_1 \quad &= \quad C_{\text{edge,max}} \left(1 + (1 + C_{\mathcal{I}})^2\right), \\ \kappa_2 \quad &= \quad C_{\text{inv}}^2 C_{\text{edge,max}} \left(1 + (1 + C_{\mathcal{I}})^2\right). \end{aligned}$$

Theorem (Global a Posteriori Error Estimate)

A global a posteriori error estimate for the energy norm is given by[1,2]

$$\|u - u_h\|_a^2 \leq \eta_1^2 + \eta_2^2 + \eta_3^2,$$

where

$$\eta_1^2 = \sum_{K \in \mathcal{T}_h} \min\left\{ \frac{4C_{\mathcal{I}}^2}{\sigma}, \frac{4C_{\mathcal{I}}^2 h_K^2}{\varepsilon} \right\} \|R_K(u_h)\|_{L^2(K)}^2,$$

$$\eta_2^2 = \sum_{F \in \mathcal{F}_h} \min\left\{ \frac{4C_F^2 h_F}{\varepsilon}, \frac{4C_F^2}{\sigma^{1/2}\varepsilon^{1/2}} \right\} \|R_F(u_h)\|_{L^2(F)}^2,$$

$$\eta_3^2 = \sum_{E \in \mathcal{E}_h} \min\left\{ \frac{4\kappa_1 h_E^2}{\varepsilon}, \frac{4\kappa_2}{\sigma} \right\} |b_E|^2 h_E^{1-d} \|\nabla u_h \cdot \mathbf{t}_E\|_{L^2(E)}^2$$

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[1] J.: *Computers & Mathematics with Applications*, 97 (**1**), 86–99, 2021

[2] J.: *Applied Mathematics Letters*, 157 (**1**), 109192, 2024

- Algebraic Flux Correction Schemes
  - Stabilization Term

$$b_h(u; v, w) = \sum_{i,j=1}^{N} (1 - \alpha_{ij}(u))\, d_{ij}\, (v(x_j) - v(x_i))\, w(x_i)$$

[1] Kuzmin: CIMNE, 2007

[2] Barrenechea, John, Knobloch: *Mathematical Models and Methods in Applied Sciences*, 27 (**3**), 525–548, 2017

[3] Kuzmin: *Computer Methods in Applied Mechanics and Engineering*, 361 (**1**), 112804, 2020

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  - $d_{ij} = \begin{cases} -\max\{a_{ij}, 0, a_{ji}\}, & i \neq j \\ -\sum_{i\neq j} d_{ij} & i = j \end{cases}$

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  - Limiters (Symmetric)
    - Kuzmin[1]
    - BJK[2]
    - Monolithic Convex[3]

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[2] Barrenechea, John, Knobloch: *Springer*, 61 (**1**), 441–447, 2025

[3] John, Knobloch: *Numerische Mathematik*, 152 (**3**), 553–585, 2022

[4] Knobloch: *Numerical Algorithms*, 94 (**2**), 547–580, 2023

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Numerical Studies

- Algebraically Stabilized Schemes
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  - Limiters
    - BBK[1] ($b_{ij} = -\gamma_0 h_{ij} \max\{\eta_i(u), \eta_j(u)\}^p$)
    - Modified-BBK (MBBK)[2] ($b_{ij} = -d_{ij} \max\{\eta_i(u), \eta_j(u)\}^p$)
    - MUAS[3]
    - SMUAS[4]

[1] Barrenechea, Burman, Karakatsani: *Numerische Mathematik*, 135 (**2**), 521–545, 2017

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  - Nonlinear Equations

- Iterative solver
  - Matrix formulation of the algebraic stabilised schemes[1,2]

$$(\mathbb{A} + \mathbb{D})\, U = F + (\mathbb{D} - \mathbb{B}(U))\, U$$

  - Fixed point right-hand side

$$\begin{aligned} (\mathbb{A} + \mathbb{D})\, \tilde{U}^{\mu} &= F + (\mathbb{D} - \mathbb{B}(U^{\mu}))\, U^{\mu}, \\ U^{\mu+1} &= \omega \tilde{U}^{\mu} + (1-\omega) U^{\mu}, \end{aligned}$$

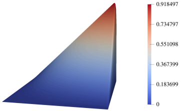
where $1 \geq \omega > 0$ is a dynamic damping parameter[3]

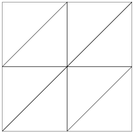
[1] J., John: *BAIL 2018*, 135 (**1**), 113–128, 2020

[2] J.,John: *Computers & Mathematics with Applications*, 78 (**9**), 3117–3138, 2019

[3] John, Knobloch: *Computer Methods in Applied Mechanics and Engineering*, 196 (**17-20**), 2197–2215, 2007

- Corner Boundary Layer[1]
  - $\Omega = (0,1)^2, \varepsilon = 10^{-2}$, $\mathbf{b} = (2,3)^\top$, $c = 1, u_D = 0, u_N = 0, \Gamma = \Gamma_D$, and f such that
  - $$u(x,y) = xy^2 - y^2 e^{\left(\frac{2(x-1)}{\varepsilon}\right)} - xe^{\left(\frac{3(y-1)}{\varepsilon}\right)} + e^{\left(\frac{2(x-1)+3(y-1)}{\varepsilon}\right)}$$





[1] John, Knobloch, Savescu: *Computer Methods in Applied Mechanics and Engineering*, 200 (**41-44**), 2916–2929, 2011

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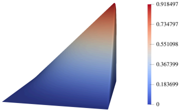

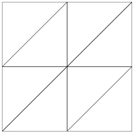

- stop of the non linear iteration [2]
    - 10000 iterations
    - $\|\text{residual}\|_{\ell^2} \leq \sqrt{\#\text{dof}}\,10^{-8}$

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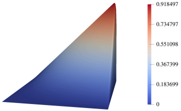
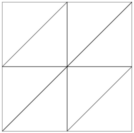
  - $$u(x,y) = xy^2 - y^2 e^{\left(\frac{2(x-1)}{\varepsilon}\right)} - xe^{\left(\frac{3(y-1)}{\varepsilon}\right)} + e^{\left(\frac{2(x-1)+3(y-1)}{\varepsilon}\right)}$$



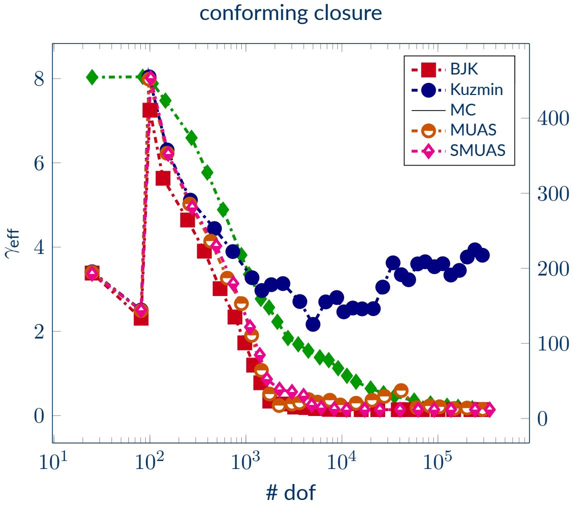


- stop of the non linear iteration [2]
  - 10000 iterations
  - $\|\text{residual}\|_{\ell^2} \leq \sqrt{\#\text{dof}}10^{-8}$
- stop of the adaptive algorithm
  - $\eta \leq 10^{-3}$
  - $\#\text{dof} \approx 2.5 \times 10^6$

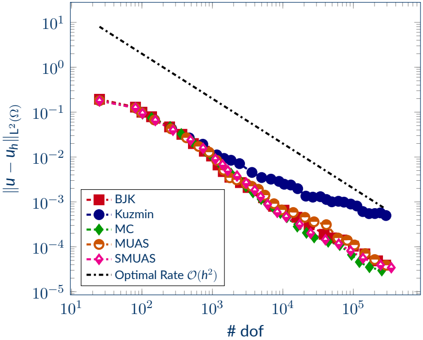
[1] John, Knobloch, Savescu: *Computer Methods in Applied Mechanics and Engineering*, 200 (**41-44**), 2916–2929, 2011

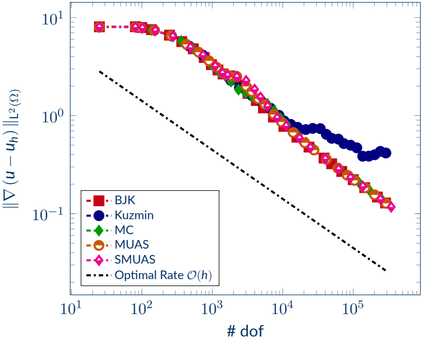
[2] J.,John: *Computers & Mathematics with Applications*, 78 (**1**), 3117–3138, 2019

- Effectivity Index



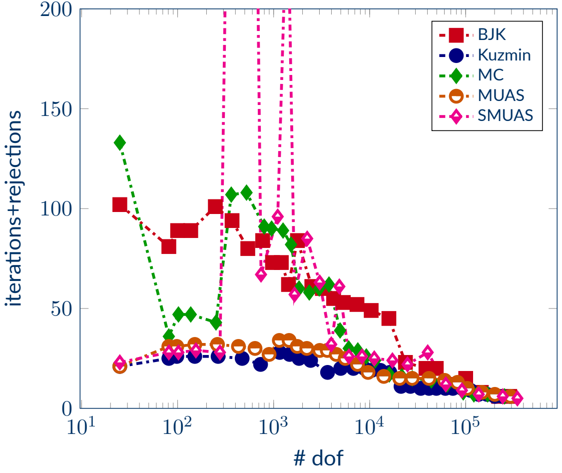
- $L^2(\Omega)$ Error and $L^2(\Omega)$ Error of the gradient[1]





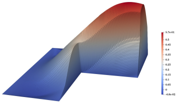
[1] Barrenechea, John, Knobloch: *SIAM Journal on Numerical Analysis*, 54 (**4**), 2427–2451, 2016

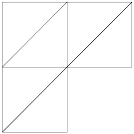
- Efficiency



200
150
100
50
0
iterations+rejections
10^1
10^2
10^3
10^4
10^5
dof
BJK
Kuzmin
MC
MUAS
SMUAS

- L-Shaped Domain[1]
  - $\Omega = (0,1)^2 \setminus (0.5,1) \times (0,0.5)$, $\varepsilon = 10^{-6}$, $\mathbf{b} = (3,1)^\top$, $c = 1$, and $\Gamma = \Gamma_D$
  - $f = 100r(r - 0.5)(r - 1/\sqrt{2})$, u_D
  - $u \notin H^2(\Omega)$





[1] John: *Computer Methods in Applied Mechanics and Engineering*, 190 (**5-7**), 757–781, 2000

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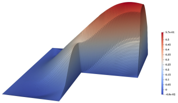

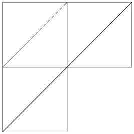

- stop of the non linear iteration
  - 10000 iterations
  - $\|\text{residual}\|_{\ell^2} \le \sqrt{\#\text{dof}}\,10^{-8}$

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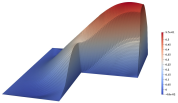

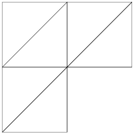

- stop of the non linear iteration
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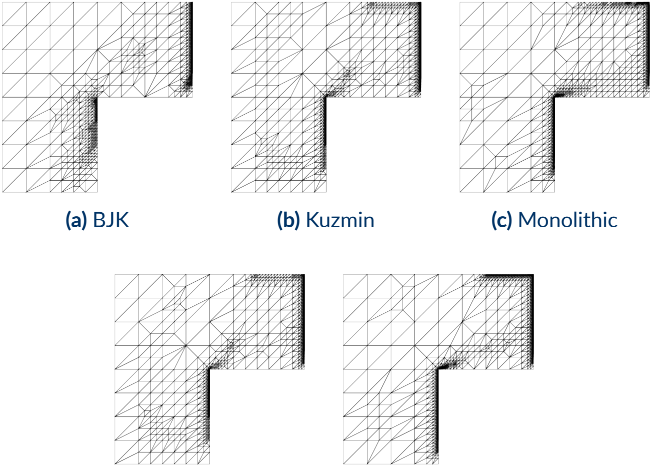
[1] John: *Computer Methods in Applied Mechanics and Engineering*, 190 (**5-7**), 757–781, 2000

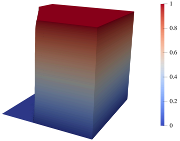
- Adaptive Grids (#dof $= 50000$)

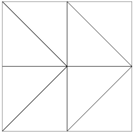


(a) BJK (b) Kuzmin (c) Monolithic

(d) MUAS (e) SMUAS

- Interior and Boundary Layer[1]
  - $\Omega = (0,1)^2$, $\varepsilon = 10^{-6}$, $\mathbf{b} = ((\cos(-\pi/3), \sin(-\pi/3))^\top$, $c = f = 0$, and $\Gamma = \Gamma_D$
  - $$u_D = \begin{cases} 1 & \text{if } (y = 1 \wedge x > 0) \text{ or } (x = 0 \wedge y > 0.7) \\ 0 & \text{else} \end{cases}$$





[1] Hughes, Mallet, Akira: *Computer Methods in Applied Mechanics and Engineering*, 54 (**3**), 341–355, 1986

Numerical Studies

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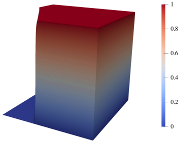

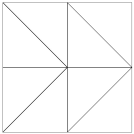

- stop of the non linear iteration
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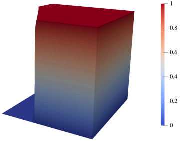

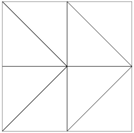

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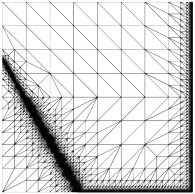
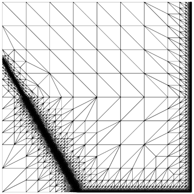
[1] Hughes, Mallet, Akira: *Computer Methods in Applied Mechanics and Engineering*, 54 (**3**), 341–355, 1986

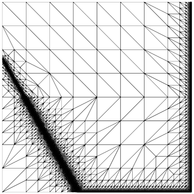
- Adaptive Grids (#dof $= 250000$)



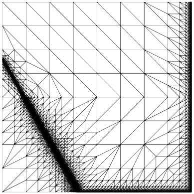
(a) BJK



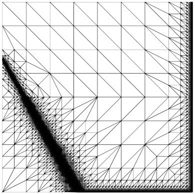
(b) Kuzmin



(c) Monolithic

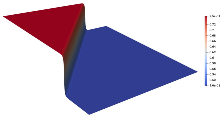


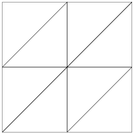
(d) MUAS



(e) SMUAS

- Nonlinear Convection[1]
  - $\Omega = (0,1)$, $\varepsilon = 10^{-3}$, $\mathbf{b} = (u,u)^\top$, $c = 0$, and $\Gamma = \Gamma_{\mathrm{D}}$
  - f and u_{D} are chosen such that
  - $$u(x,y) = \frac{3}{4} - \frac{1}{4\left[1 + \exp\left(\frac{-4x+4y-1}{32\varepsilon}\right)\right]}$$





[1] Zhao, Li, Gao: *Journal of Mathematical Analysis and Applications*, 550 (**2**), 129617, 2025

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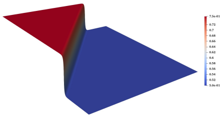

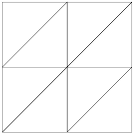

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Numerical Studies

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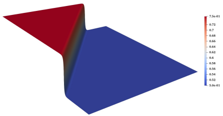

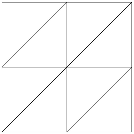

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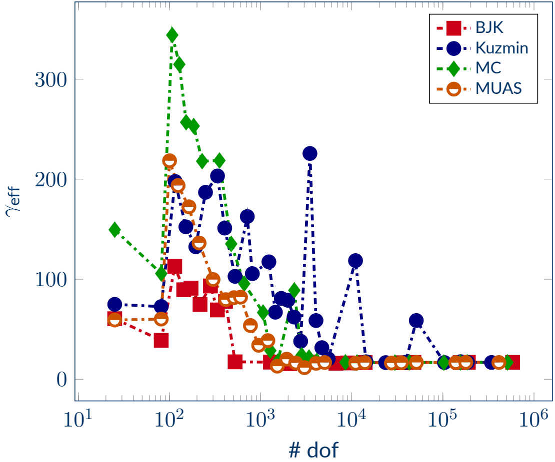
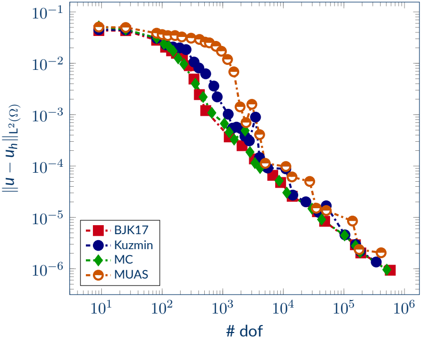
[1] Zhao, Li, Gao: *Journal of Mathematical Analysis and Applications*, 550 (**2**), 129617, 2025

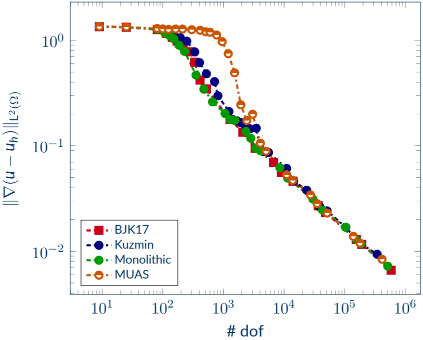
- Effectivity Index



Numerical Studies

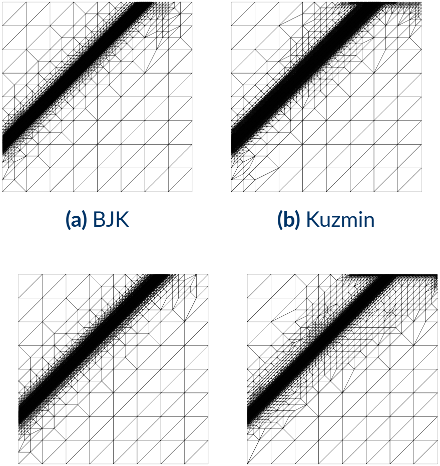
- $L^2(\Omega)$ Error and $L^2(\Omega)$ Error of the gradient





Numerical Studies

- Adaptive Grids (#dof $= 500000$)



(a) BJK (b) Kuzmin (c) Monolithic (d) MUAS

- Conclusions
  - Linear Example

	BJK	Kuzmin	MC	MUAS	SMUAS
γ_{eff}	$\circ$	$-$	$++$	$\circ$	$\circ$
Error	$++$	$-$	$++$	$++$	$++$
Efficiency	$-$	$++$	$+$	$++$	$\circ$
Adaptive Grids	$\circ$	$++$	$++$	$++$	$+$

Conclusions

- Conclusions
  - Linear Example

	BJK	Kuzmin	MC	MUAS	SMUAS
γ_{eff}	$\circ$	$-$	$++$	$\circ$	$\circ$
Error	$++$	$-$	$++$	$++$	$++$
Efficiency	$-$	$++$	$+$	$++$	$\circ$
Adaptive Grids	$\circ$	$++$	$++$	$++$	$+$

  - Nonlinear Example

	BJK	Kuzmin	MC	MUAS
γ_{eff}	$++$	$-$	$-$	$+$
Error	$++$	$-$	$++$	$-$
Adaptive Grids	$++$	$-$	$\circ$	$+$

- Summary
  - AFC+Monolithic well suited for Linear example
  - AFC+BJK well suited for Nonlinear example

Summary & Outlook

- Summary
  - AFC+Monolithic well suited for Linear example
  - AFC+BJK well suited for Nonlinear example
  - AS+MUAS good alternative

Summary & Outlook

- Summary
  - AFC+Monolithic well suited for Linear example
  - AFC+BJK well suited for Nonlinear example
  - AS+MUAS good alternative
- Outlook
  - More examples required to better understand AFC+Adaptive Grids
  - Incorporate BBK and MBBK
  - Efficient implementation for SMUAS

Thank You!

Webpage: tinyurl.com/abhi0207