# Adaptive Grids for Algebraic Stabilizations of Convection-Diffusion-Reaction Equations

Abhinav Jha

Institute of Applied Analysis and Numerical Simulation
Universität Stuttgart

SIAM Conference on Computational Science and Engineering (CSE23)
1$^{st}$ March 2023

Joint work with Volker John (WIAS, Berlin) and Petr Knobloch (Charles University, Prague)

# Outline

- Steady-state convection-diffusion-reaction equation

$$\begin{aligned} -\varepsilon\Delta u + \mathbf{b}\cdot\nabla u + cu &= f && \text{in } \Omega,\\ u &= u_b && \text{on } \Gamma_{\mathrm{D}},\\ -\varepsilon\nabla u\cdot\mathbf{n} &= g && \text{on } \Gamma_{\mathrm{N}} \end{aligned}$$

  - $\Omega$ – bounded polyhedral Lipschitz domain in $\mathbb{R}^d$, $d \in \{2,3\}$
  - $\mathbf{n}$ – outward pointing unit normal
  - Assume
$$\left(c(x) - \frac{1}{2}\nabla\cdot\mathbf{b}(x)\right) \geq \sigma_0 > 0$$
  - Interested in convection-dominated regime, $\varepsilon \ll \|\mathbf{b}\|_{L^\infty(\Omega)} L$
  - $L$ – Characteristic length of the problem

- Ideal discretization
  1. Accurate and sharp layers

- Ideal discretization
    1. Accurate and sharp layers
    2. Physically consistent results (no spurious oscillations)

- Ideal discretization
    1. Accurate and sharp layers
    2. Physically consistent results (no spurious oscillations)
    3. Efficient computation of the solutions

- Ideal discretization
  1. Accurate and sharp layers
  2. Physically consistent results (no spurious oscillations)
  3. Efficient computation of the solutions
- Because of $2^{\text{nd}}$ property: Algebraic stabilised schemes very well suited for applications

- Ideal discretization
  1. Accurate and sharp layers
  2. Physically consistent results (no spurious oscillations)
  3. Efficient computation of the solutions
- Because of $2^{\text{nd}}$ property: Algebraic stabilised schemes very well suited for applications
- Alternate approach: Adaptive grids

- Ideal discretization
  1. Accurate and sharp layers
  2. Physically consistent results (no spurious oscillations)
  3. Efficient computation of the solutions
- Because of $2^{\text{nd}}$ property: Algebraic stabilised schemes very well suited for applications
- Alternate approach: Adaptive grids
- Idea: Combine both the approaches

- Variational problem for AFC scheme[1]
  Find $u_h \in V_h$ such that

$$a_h(u_h, v_h) + d_h(u_h; u_h, v_h) = \langle f, v_h \rangle \quad \forall\, v_h \in V_h$$

  - $V_h-$ finite element space with homogeneous Dirichlet boundary conditions $(V_h \subset V)$

[1] Barrenechea, John, Knobloch: arXiv : 2204.07480, 2022

- Variational problem for AFC scheme[1]
  Find $u_h \in V_h$ such that

$$a_h(u_h, v_h) + d_h(u_h; u_h, v_h) = \langle f, v_h \rangle \quad \forall\, v_h \in V_h$$

  - $V_h -$ finite element space with homogeneous Dirichlet boundary conditions $(V_h \subset V)$
  - stabilization

$$d_h(w; z, v) = \sum_{i,j=1}^{N} (1 - \alpha_{ij}(w)) d_{ij}(z_j - z_i) v_i \quad \forall\, w, v, z \in V_h$$

[1] Barrenechea, John, Knobloch: arXiv : 2204.07480, 2022

- Variational problem for AFC scheme[1]
  Find $u_h \in V_h$ such that

$$a_h(u_h, v_h) + d_h(u_h; u_h, v_h) = \langle f, v_h \rangle \quad \forall\, v_h \in V_h$$

  - $V_h -$ finite element space with homogeneous Dirichlet boundary conditions $(V_h \subset V)$
  - stabilization

$$d_h(w; z, v) = \sum_{i,j=1}^{N} (1 - \alpha_{ij}(w)) d_{ij}(z_j - z_i) v_i \quad \forall\, w, v, z \in V_h$$

- Energy norm

$$\|u_h\|_a^2 = \varepsilon |u_h|_1^2 + \sigma \|u_h\|_0^2$$

[1] Barrenechea, John, Knobloch: arXiv : 2204.07480, 2022

## Theorem (Global a posteriori error estimate)

*A global a posteriori error estimate for the energy norm is given by*[1]

$$\|u - u_h\|_a^2 \leq \eta_1^2 + \eta_2^2 + \eta_3^2,$$

*where*

$$\eta_1^2 = \sum_{K \in \mathcal{T}_h} \min\left\{\frac{4C_I^2}{\sigma}, \frac{4C_I^2 h_K^2}{\varepsilon}\right\} \|R_K(u_h)\|_{L^2(K)}^2,$$

$$\eta_2^2 = \sum_{F \in \mathcal{F}_h} \min\left\{\frac{4C_F^2 h_F}{\varepsilon}, \frac{4C_F^2}{\sigma^{1/2}\varepsilon^{1/2}}\right\} \|R_F(u_h)\|_{L^2(F)}^2,$$

$$\eta_3^2 = \sum_{E \in \mathcal{E}_h} \min\left\{\frac{4\kappa_1 h_E^2}{\varepsilon}, \frac{4\kappa_2}{\sigma}\right\} (1 - \alpha_E)^2 |d_E|^2 h_E^{1-d} \|\nabla u_h \cdot \mathbf{t}_E\|_{L^2(E)}^2$$

[1] J.: CAMWA, 97(1), 86–99, 2021

- Standard strategy for solving

**SOLVE $\rightarrow$ ESTIMATE $\rightarrow$ MARK $\rightarrow$ REFINE**

[1] Xu, Zikatanov: MC, 68(228), 1429–1446, 1999

# Hanging Nodes

- Standard strategy for solving

**SOLVE $\rightarrow$ ESTIMATE $\rightarrow$ MARK $\rightarrow$ REFINE**

- Hanging nodes
  - Preserves angles after red-refinement
  - Avoids prism and pyramids in 3D mesh refinement
  - *hp* adaptive refinement
- Certain stabilized schemes rely on the property of triangulation [1]

[1] Xu, Zikatanov: MC, 68(228), 1429–1446, 1999

## Lemma

*Let $\mathcal{T}$ be a non-conforming triangulation of $\Omega$, i.e., $\mathcal{T}$ has hanging nodes. Then, for all $q \in H(\mathcal{T})$ there are coefficients $a_{qp}$ with $p \in N_F(\mathcal{T}) \setminus H(\mathcal{T})$ such that all $v \in V_h$ can be represented as*[1,2]

$$v(q) = \sum_{p \in N_F(\mathcal{T}) \setminus H(\mathcal{T})} a_{qp} v(p)$$

[1] Gräser : PhD Thesis, FU Berlin 2011

[2] J.: PhD Thesis, FU Berlin 2020

## Lemma

*Let $\mathcal{T}$ be a non-conforming triangulation of $\Omega$, i.e., $\mathcal{T}$ has hanging nodes. Then, for all $q \in H(\mathcal{T})$ there are coefficients $a_{qp}$ with $p \in N_F(\mathcal{T}) \setminus H(\mathcal{T})$ such that all $v \in V_h$ can be represented as*[1,2]

$$v(q) = \sum_{p \in N_F(\mathcal{T}) \setminus H(\mathcal{T})} a_{qp} v(p)$$

## Theorem

*Let $\{\mathcal{T}_0, \cdots, \mathcal{T}_j\}$ be a grid hierarchy on $\Omega$ with $\mathcal{T}_0$ being conforming. Let us denote $\mathcal{T} = \mathcal{T}_j$, i.e., the final refinement level. Then a basis of $V_h$ is given by*[1]

$$B(\mathcal{T}) := \left\{ \varphi_p = \varphi_p^{\mathrm{nc}} + \sum_{q \in H(\mathcal{T})} a_{qp} \varphi_q^{\mathrm{nc}} : p \in N_F(\mathcal{T}) \setminus H(\mathcal{T}) \right\}$$

[1] Gräser : PhD Thesis, FU Berlin 2011

[2] J.: PhD Thesis, FU Berlin 2020

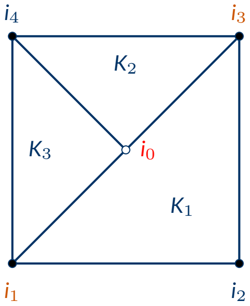
- Satisfaction of DMP
  - DMP is satisfied if[1]

$$\begin{aligned} a_{ii} &> 0, \\ a_{ij} + a_{ji} &\leq 0, \end{aligned}$$

where $a_{ij}$ is in the stiffness matrix

[1] Barrenechea, John, Knobloch: SINUM (54), 2427–2451, 2016

- Consider the sample patch



- Initial assembly

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \\ a_{20} & a_{21} & a_{22} & a_{23} & a_{24} \\ a_{30} & a_{31} & a_{32} & a_{33} & a_{34} \\ a_{40} & a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}, \quad \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

- Initial assembly

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} \\ a_{20} & a_{21} & a_{22} & a_{23} & a_{24} \\ a_{30} & a_{31} & a_{32} & a_{33} & a_{34} \\ a_{40} & a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}, \quad \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

- Conforming test space and continuity of the hanging node

$$\begin{pmatrix} 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ a_{10} + \frac{a_{00}}{2} & a_{11} + \frac{a_{01}}{2} & a_{12} + \frac{a_{02}}{2} & a_{13} + \frac{a_{03}}{2} & a_{14} + \frac{a_{04}}{2} \\ a_{20} & a_{21} & a_{22} & a_{23} & a_{24} \\ a_{30} + \frac{a_{00}}{2} & a_{31} + \frac{a_{01}}{2} & a_{32} + \frac{a_{02}}{2} & a_{33} + \frac{a_{03}}{2} & a_{34} + \frac{a_{04}}{2} \\ a_{40} & a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}, \quad \begin{pmatrix} 0 \\ b_1 + \frac{b_0}{2} \\ b_2 \\ b_3 + \frac{b_0}{2} \\ b_4 \end{pmatrix}$$

- Conforming ansatz space

$$\begin{pmatrix} 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & a_{11}+\frac{a_{01}}{2}+\frac{a_{10}}{2}+\frac{a_{00}}{4} & a_{12}+\frac{a_{02}}{2} & a_{13}+\frac{a_{03}}{2}+\frac{a_{10}}{2}+\frac{a_{00}}{4} & a_{14}+\frac{a_{04}}{2} \\ 0 & a_{21}+\frac{a_{20}}{2} & a_{22} & a_{23}+\frac{a_{20}}{2} & a_{24} \\ 0 & a_{31}+\frac{a_{01}}{2}+\frac{a_{30}}{2}+\frac{a_{00}}{4} & a_{32}+\frac{a_{02}}{2} & a_{33}+\frac{a_{03}}{2}+\frac{a_{30}}{2}+\frac{a_{00}}{4} & a_{34}+\frac{a_{04}}{2} \\ 0 & a_{41}+\frac{a_{40}}{2} & a_{42} & a_{43}+\frac{a_{40}}{2} & a_{44} \end{pmatrix}$$

- Conforming ansatz space

$$
\begin{pmatrix}
1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\
0 & a_{11}+\frac{a_{01}}{2}+\frac{a_{10}}{2}+\frac{a_{00}}{4} & a_{12}+\frac{a_{02}}{2} & a_{13}+\frac{a_{03}}{2}+\frac{a_{10}}{2}+\frac{a_{00}}{4} & a_{14}+\frac{a_{04}}{2} \\
0 & a_{21}+\frac{a_{20}}{2} & a_{22} & a_{23}+\frac{a_{20}}{2} & a_{24} \\
0 & a_{31}+\frac{a_{01}}{2}+\frac{a_{30}}{2}+\frac{a_{00}}{4} & a_{32}+\frac{a_{02}}{2} & a_{33}+\frac{a_{03}}{2}+\frac{a_{30}}{2}+\frac{a_{00}}{4} & a_{34}+\frac{a_{04}}{2} \\
0 & a_{41}+\frac{a_{40}}{2} & a_{42} & a_{43}+\frac{a_{40}}{2} & a_{44}
\end{pmatrix}
$$

- Increases the matrix stencil by few elements

- Algebraic stabilisation schemes
  - Algebraic Flux Correction (AFC) schemes
    - Kuzmin limiter[1]
    - BJK limiter[2,3]

[1] Kuzmin: in Proc. Int. Conf. Comput. Meth. for Coupled Problems in Science and Engineering, CIMNE, 2007

[2] Barrenechea, John, Knobloch: M3AS (27), 525–548, 2017

[3] Barrenechea, John, Knobloch: arXiv: 2204.07480, 2022

[4] John, Knobloch: arXiv: 2111.08697, 2021

- Algebraic stabilisation schemes
  - Algebraic Flux Correction (AFC) schemes
    - Kuzmin limiter[1]
    - BJK limiter[2,3]
  - Monotone Upwind-type Algebraically Stabilized (MUAS) method[4]
    - Drops symmetric condition on $\alpha_{ij}$
    - AFC system is modified

$$(A + D)\, U = F + (D - B)\, U,$$

where

$$b_{ij} = \max\left\{\left(1 - \overline{\alpha_{ij}}(u)\right) a_{ij}, 0, \left(1 - \overline{\alpha_{ji}}(u)\right) a_{ji}\right\}$$

[1] Kuzmin: in Proc. Int. Conf. Comput. Meth. for Coupled Problems in Science and Engineering, CIMNE, 2007

[2] Barrenechea, John, Knobloch: M3AS (27), 525–548, 2017

[3] Barrenechea, John, Knobloch: arXiv: 2204.07480, 2022

[4] John, Knobloch: arXiv: 2111.08697, 2021

- Comparison of results:
  - Accuracy of solution
    - $\|\cdot\|_{L^2(\Omega)}$
    - $\|\nabla(\cdot)\|_{L^2(\Omega)}$

[1] Augustin, Caiazzo, Fiebach, Fuhrmann, John, Linke, Umla : CMAME (200), 3395 - 3409, 2011

- Comparison of results:
  - Accuracy of solution
    - $\| \cdot \|_{L^2(\Omega)}$
    - $\|\nabla(\cdot)\|_{L^2(\Omega)}$
  - Efficiency of the scheme

[1] Augustin, Caiazzo, Fiebach, Fuhrmann, John, Linke, Umla : CMAME (200), 3395 - 3409, 2011

- Comparison of results:
  - Accuracy of solution
    - $\|\cdot\|_{L^2(\Omega)}$
    - $\|\nabla(\cdot)\|_{L^2(\Omega)}$
  - Efficiency of the scheme
  - Global satisfaction of DMP

$$\mathrm{osc}_{\max}(u_h) := \max_{(x,y)\in\overline{\Omega}} u_h(x,y) - 1 - \min_{(x,y)\in\overline{\Omega}} u_h(x,y)$$

[1] Augustin, Caiazzo, Fiebach, Fuhrmann, John, Linke, Umla : CMAME (200), 3395 - 3409, 2011

- Comparison of results:
  - Accuracy of solution
    - $\|\cdot\|_{L^2(\Omega)}$
    - $\|\nabla(\cdot)\|_{L^2(\Omega)}$
  - Efficiency of the scheme
  - Global satisfaction of DMP

$$\mathrm{osc}_{\max}(u_h) := \max_{(x,y)\in\overline{\Omega}} u_h(x,y) - 1 - \min_{(x,y)\in\overline{\Omega}} u_h(x,y)$$

  - Smearing of internal layer[1]

[1] Augustin, Caiazzo, Fiebach, Fuhrmann, John, Linke, Umla : CMAME (200), 3395 - 3409, 2011

- Comparison of results:
  - Accuracy of solution
    - $\|\cdot\|_{L^2(\Omega)}$
    - $\|\nabla(\cdot)\|_{L^2(\Omega)}$
  - Efficiency of the scheme
  - Global satisfaction of DMP

$$\mathrm{osc}_{\max}(u_h) := \max_{(x,y)\in\overline{\Omega}} u_h(x,y) - 1 - \min_{(x,y)\in\overline{\Omega}} u_h(x,y)$$

  - Smearing of internal layer[1]
  - Adaptive grids

[1] Augustin, Caiazzo, Fiebach, Fuhrmann, John, Linke, Umla : CMAME (200), 3395 - 3409, 2011

- Comparison of results:
  - Accuracy of solution
    - $\|\cdot\|_{L^2(\Omega)}$
    - $\|\nabla(\cdot)\|_{L^2(\Omega)}$
  - Efficiency of the scheme
  - Global satisfaction of DMP

$$\mathrm{osc}_{\max}(u_h) := \max_{(x,y)\in\overline{\Omega}} u_h(x,y) - 1 - \min_{(x,y)\in\overline{\Omega}} u_h(x,y)$$

  - Smearing of internal layer[1]
  - Adaptive grids
    - Conforming closure
    - Hanging nodes

[1] Augustin, Caiazzo, Fiebach, Fuhrmann, John, Linke, Umla : CMAME (200), 3395 - 3409, 2011

- Comparison of results:
  - Accuracy of solution
    - $\| \cdot \|_{L^2(\Omega)}$
    - $\|\nabla(\cdot)\|_{L^2(\Omega)}$
  - Efficiency of the scheme
  - Global satisfaction of DMP

$$\mathrm{osc}_{\max}(u_h) := \max_{(x,y)\in\overline{\Omega}} u_h(x,y) - 1 - \min_{(x,y)\in\overline{\Omega}} u_h(x,y)$$

  - Smearing of internal layer[1]
  - Adaptive grids
    - Conforming closure
    - Hanging nodes
- For MUAS method neglect $\eta_3$

[1] Augustin, Caiazzo, Fiebach, Fuhrmann, John, Linke, Umla : CMAME (200), 3395 - 3409, 2011

- Iterative solver
  - Matrix formulation of the algebraic stabilised schemes[1,2]

$$(A + D)\,U = F + (D - B(U))\,U$$

  - Fixed point right-hand side

$$\begin{aligned}(A + D)\,\tilde{U}^{\mu} &= F + (D - B(U^{\mu}))\,U^{\mu},\\ U^{\mu+1} &= \omega\tilde{U}^{\mu} + (1-\omega)U^{\mu},\end{aligned}$$

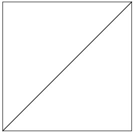
where $\omega > 0$ is a dynamic damping parameter

[1] J.John: BAIL 2018 (135), 2020

[2] J.,John: CAMWA (78), 3117-3138, 2019

- Example with corner boundary layer[1]
- $\Omega = (0,1)^2, \varepsilon = 10^{-2}, \mathbf{b} = (2,3)^T, c = 1, u_b = 0, g = 0$, and $f$ such that

$$u(x,y) = xy^2 - y^2 \exp\left(\frac{2(x-1)}{\varepsilon}\right) - x\exp\left(\frac{3(y-1)}{\varepsilon}\right) + \exp\left(\frac{2(x-1)+3(y-1)}{\varepsilon}\right)$$

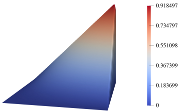
- stop of the non linear iteration[2]
  - $10000$ iterations
  - $\|\text{residual}\|_2 \leq \sqrt{\#\text{dof}}\,10^{-10}$

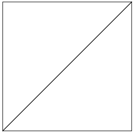
[1] John, Knobloch, Savescu: CMAME (200), 2916–2929, 2011

[2] J.,John: CAMWA (78), 3117-3138, 2019

- Example with corner boundary layer[1]
- $\Omega = (0,1)^2$, $\varepsilon = 10^{-2}$, $\mathbf{b} = (2,3)^T$, $c = 1$, $u_b = 0$, $g = 0$, and $f$ such that

$$u(x,y) = xy^2 - y^2 \exp\left(\frac{2(x-1)}{\varepsilon}\right) - x\exp\left(\frac{3(y-1)}{\varepsilon}\right) + \exp\left(\frac{2(x-1)+3(y-1)}{\varepsilon}\right)$$

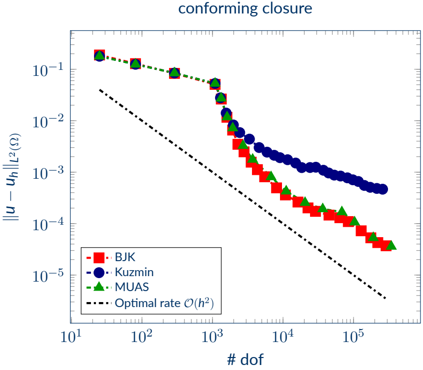


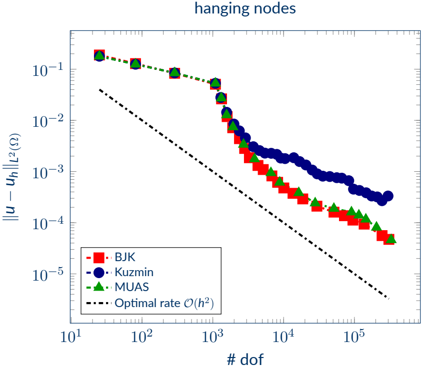



- stop of the non linear iteration [2]
  - $10000$ iterations
  - $\|\text{residual}\|_2 \leq \sqrt{\#\text{dof}}\,10^{-10}$
- stop of the adaptive algorithm
  - $\eta \leq 10^{-3}$
  - $\#\text{dof} \approx 10^6$

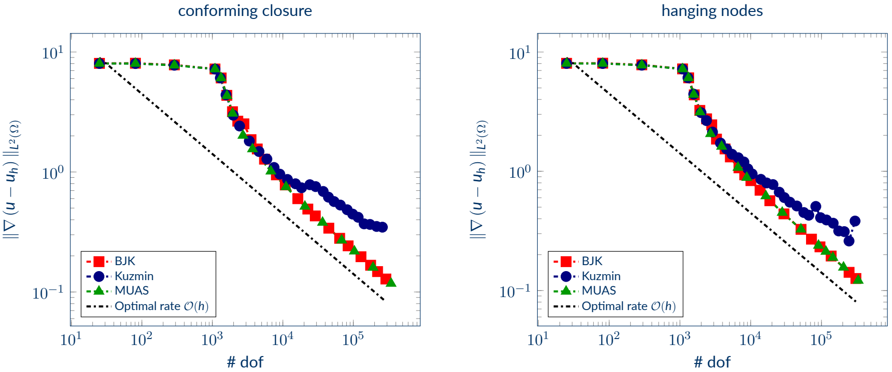
[1] John, Knobloch, Savescu: CMAME (200), 2916–2929, 2011
[2] J.,John: CAMWA (78), 3117-3138, 2019

- $L^2(\Omega)$ Error

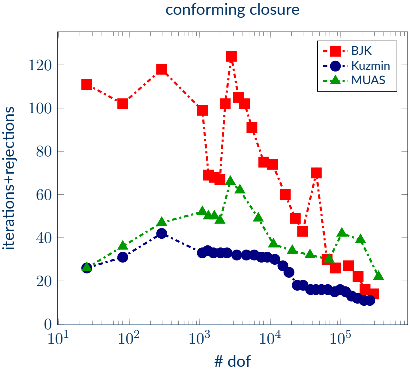


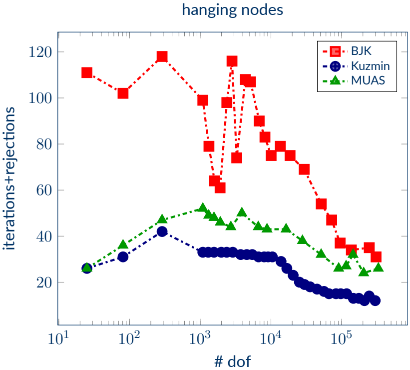



- $L^2(\Omega)$ Error of the gradient

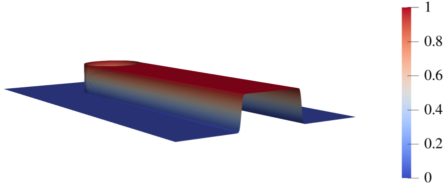


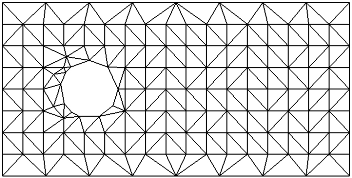
- Efficiency


conforming closure
iterations+rejections
# dof
BJK
Kuzmin
MUAS


hanging nodes
iterations+rejections
# dof
BJK
Kuzmin
MUAS

- Hemker problem[1]
- $\varepsilon = 10^{-4}$, $\mathbf{b} = (1,0)^T$, $c = f = 0$

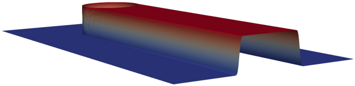


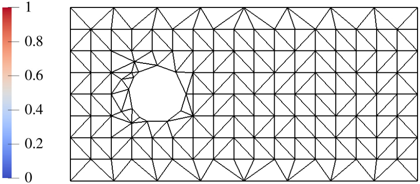



- stop of the non linear iteration
  - $10000$ iterations
  - $\|\text{residual}\|_2 \leq \sqrt{\#\text{dof}}\,10^{-8}$

[1] Hemker: JCAM 76, 277-285, 1996

- Hemker problem[1]
- $\varepsilon = 10^{-4}$, $\mathbf{b} = (1, 0)^T$, $c = f = 0$

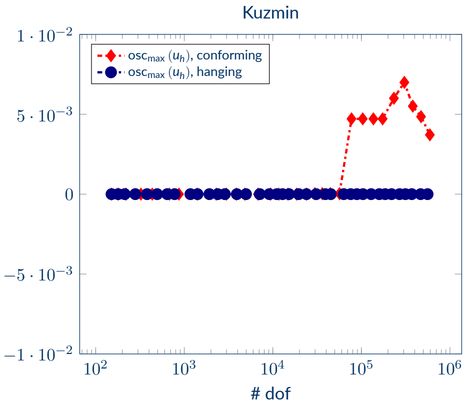




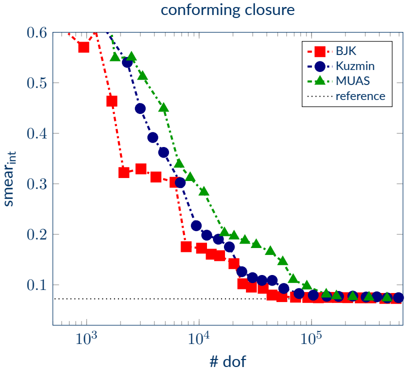

- stop of the non linear iteration
  - 10000 iterations
  - $\|\text{residual}\|_2 \leq \sqrt{\#\text{dof}}\,10^{-8}$
- stop of the adaptive algorithm
  - $\eta \leq 10^{-3}$
  - $\#\text{dof} \approx 5 \times 10^5$

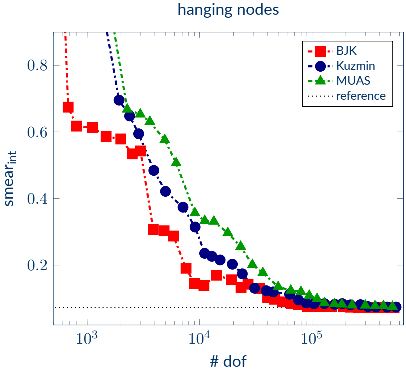
[1] Hemker: JCAM 76, 277-285, 1996

- Satisfaction of Global DMP



- Smearing of internal layer



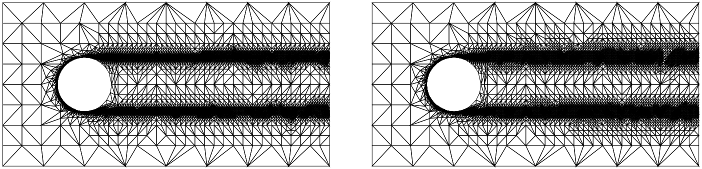
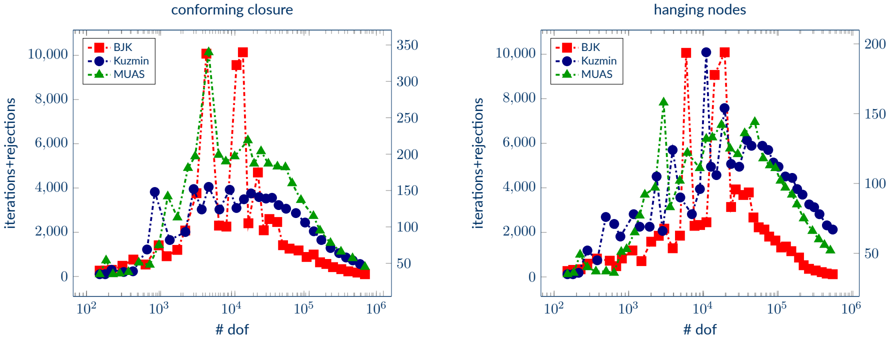


**Figure 1:** Adaptively refined conforming grids with $\approx 25,000$ #dof, AFC method and Kuzmin limiter (left), MUAS method (right)

- Efficiency


conforming closure
hanging nodes
BJK
Kuzmin
MUAS
iterations+rejections
# dof
0
2,000
4,000
6,000
8,000
10,000
50
100
150
200
250
300
350
$10^2$
$10^3$
$10^4$
$10^5$
$10^6$

- Conclusions[1]
  - Accuracy of solution
    - AFC + BJK limiter and MUAS method converge on all grids
    - AFC + Kuzmin limiter does not converge on adaptively refined grids if solution becomes (locally) diffusion-dominated

[1] J.,John, Knobloch: arXiv : 2007.08405 , 2022

[2] J.,John: CAMWA (78), 3117-3138, 2019

- Conclusions[1]
  - Accuracy of solution
    - AFC + BJK limiter and MUAS method converge on all grids
    - AFC + Kuzmin limiter does not converge on adaptively refined grids if solution becomes (locally) diffusion-dominated
  - Efficiency
    - AFC+ Kuzmin limiter and the MUAS method[2] most efficient

[1] J.,John, Knobloch: arXiv : 2007.08405 , 2022
[2] J.,John: CAMWA (78), 3117-3138, 2019

- Conclusions[1]
  - Accuracy of solution
    - AFC + BJK limiter and MUAS method converge on all grids
    - AFC + Kuzmin limiter does not converge on adaptively refined grids if solution becomes (locally) diffusion-dominated
  - Efficiency
    - AFC+ Kuzmin limiter and the MUAS method[2] most efficient
  - Satisfaction of DMP
    - Global DMP satisfied on grids with hanging nodes
    - AFC+ Kuzmin limiter did not satisfy on conformally closed grids

[1] J.,John, Knobloch: arXiv : 2007.08405 , 2022
[2] J.,John: CAMWA (78), 3117-3138, 2019

# Conclusions and Outlook



- Conclusions[1]
  - Accuracy of solution
    - AFC + BJK limiter and MUAS method converge on all grids
    - AFC + Kuzmin limiter does not converge on adaptively refined grids if solution becomes (locally) diffusion-dominated
  - Efficiency
    - AFC+ Kuzmin limiter and the MUAS method[2] most efficient
  - Satisfaction of DMP
    - Global DMP satisfied on grids with hanging nodes
    - AFC+ Kuzmin limiter did not satisfy on conformally closed grids
  - Smearing
    - AFC + BJK limiter sharpest layer
    - For fine grids, all values close to reference value

[1] J.,John, Knobloch: arXiv : 2007.08405 , 2022
[2] J.,John: CAMWA (78), 3117-3138, 2019

# Conclusions and Outlook



- Conclusions[1]
  - Accuracy of solution
    - AFC + BJK limiter and MUAS method converge on all grids
    - AFC + Kuzmin limiter does not converge on adaptively refined grids if solution becomes (locally) diffusion-dominated
  - Efficiency
    - AFC+ Kuzmin limiter and the MUAS method[2] most efficient
  - Satisfaction of DMP
    - Global DMP satisfied on grids with hanging nodes
    - AFC+ Kuzmin limiter did not satisfy on conformally closed grids
  - Smearing
    - AFC + BJK limiter sharpest layer
    - For fine grids, all values close to reference value
- MUAS method most promising

[1] J.,John, Knobloch: arXiv : 2007.08405 , 2022

[2] J.,John: CAMWA (78), 3117-3138, 2019

- Outlook
  - Development of estimators for MUAS method
  - Numerical studies in 3D
  - Comparison with Monolithic Convex Limiter[1][2]

[1] Kuzmin: CMAME (361), 112804, 2020

[2] J., Partl, Ahmed, Kuzmin: JNUM, 10.1515/jnma-2021-0123, 2022